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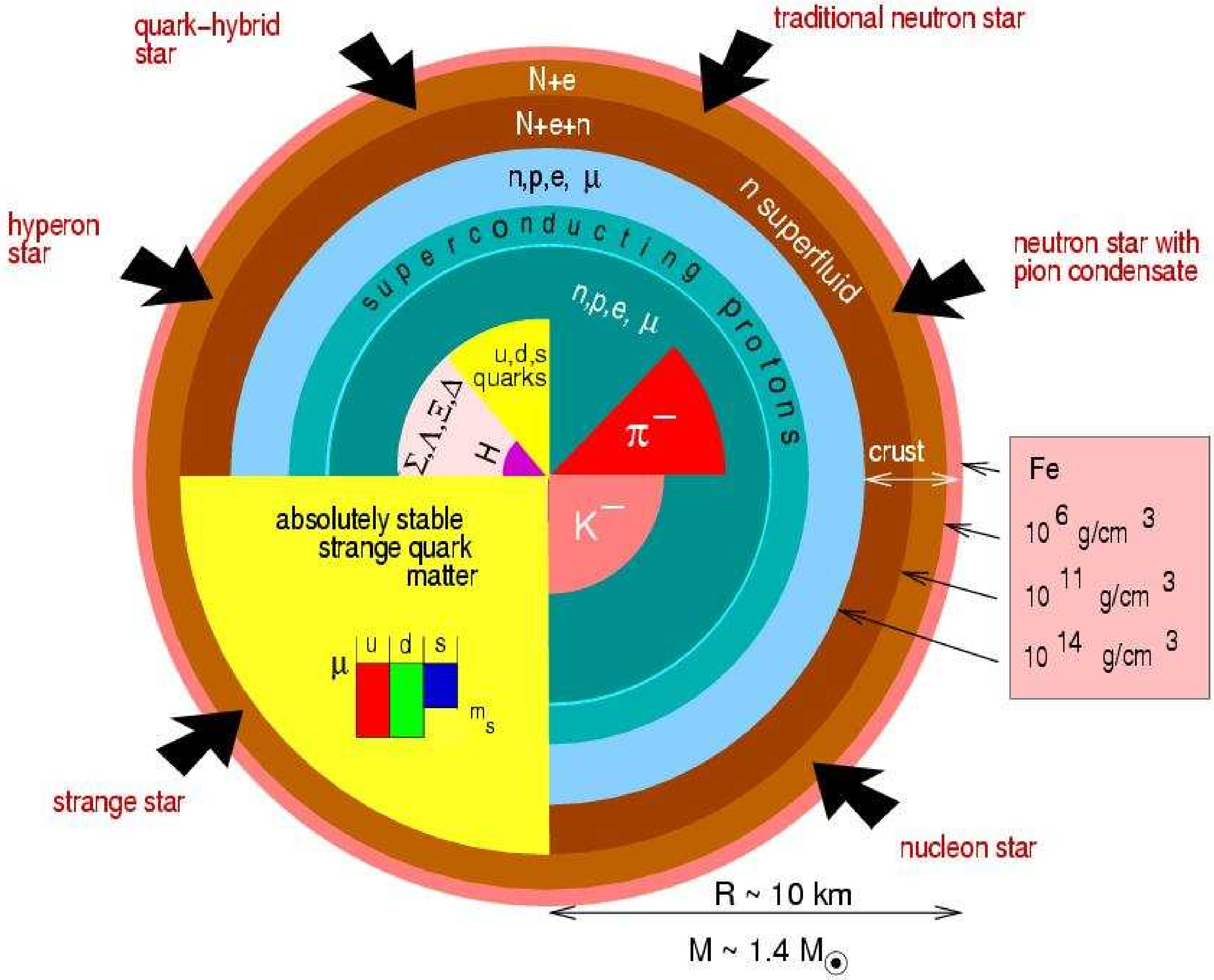
Spin dynamics



Physics on three different scales

- Micro-physics (Pair size) $L_1 \sim \xi,\ \lambda \sim$ 10-100 fm
- Mesoscopic physics (vortex size) $L_2 \sim d_n \sim$ 10^{-3} cm
- Macrophysics (star size) $L_3 \sim R \sim$ km
- There is a of scales hierarchy $L_1 \ll L_2 \ll L_3$

Todays topic is the relation

$$\text{Meso} \leftrightarrow \text{Macro} \leftrightarrow \text{Observations}$$

from compact star rotational dynamics.

Microscale physics

- At low temperatures $T \leq T_c \sim 10^9$ K neutrons and protons form Cooper pairs (super-fluidity, -conductivity).
- Low density/energy order parameter ($^1S_0$ partial wave)

$$\Delta = -i\sigma_y \hat{1} \langle \psi(x)\psi(x')\rangle = -i\sigma_y \hat{1} \langle \psi^\dagger(x)\psi^\dagger(x')\rangle \sim 1 \text{ MeV} \tag{1}$$

- High density/energy order parameter ($^3P_2$ partial wave)

$$\Delta = \sigma_z \hat{1} \langle \psi(x)\psi(x')\rangle = \sigma_z \hat{1} \langle \psi^\dagger(x)\psi^\dagger(x')\rangle \sim 0.1 \text{ MeV} \tag{2}$$

- Color superconductivity - 2SC, LOFF, CFL etc: gaps are by 1-2 orders of magnitude larger

Meso-scale physics

- Long-wave length behavior is described by two fluid hydrodynamics: (one-component neutral fluid)

$$\rho = \rho_S + \rho_N, \quad \vec{v}_S, \quad \vec{v}_N \quad \text{(independent)} \tag{3}$$

- Superfluid is characterized by a single wave-function

$$\psi(\vec{x}) = f(x)e^{i\theta}, \quad |\psi(\vec{x})|^2 = \rho_S, \quad \vec{v}_S = \frac{\hbar}{m}\vec{\nabla}\theta(x) \tag{4}$$

- Superfluid flow is irrotational

$$\vec{\nabla} \times \vec{v}_S = 0, \tag{5}$$

- except when rotated with $\Omega \geq \Omega_{c1}$, quantized circulation!

$$\vec{\nabla} \times \vec{v}_S = \frac{2\pi\hbar}{m}, \qquad \int \vec{v}_S \cdot \, d\vec{l} = \frac{2\pi\hbar}{m}. \tag{6}$$

- Concept of vortex (Feynman-Onsager, Abrikosov)
  Ginzburg-Landau eq. for $f(r)$ lead to:

$$\frac{d^2 f}{d\zeta^2} + \frac{1}{\zeta}\frac{df}{d\zeta} - \frac{1}{\zeta^2} f + f - f^3 = 0, \quad f(\zeta) = \begin{cases} C\zeta & \zeta \ll 1, \\ 1 - (2\zeta^2)^{-1} & \zeta \to \infty, \end{cases} \tag{7}$$

  where $\zeta = x/\xi_n$ and ξ_n is the size of the vortex core.

- Vortex velocity (axially symmetric solution)

$$v_S = \frac{\hbar}{mr}\hat{\phi}. \tag{8}$$

Including electro-magnetism

- gauge invariant superfluid velocities

$$\vec{v}_S = \frac{\hbar}{m}\nabla\theta - \frac{2e}{mc}\vec{A}, \tag{9}$$

- Vortex solutions

$$\vec{\nabla} \times \vec{v}_S = \frac{\pi\hbar}{m}\vec{\nu}_S \sum_j \delta^{(2)}(\vec{x} - \vec{x}_p) - \frac{2e}{mc}\vec{B} \equiv \vec{\omega}_S, \tag{10}$$

- From $\vec{\nabla} \times \vec{B} = (4\pi/c)\vec{j}_S$ and $\vec{j}_S = ne\vec{v}_S$ follows the London equation

$$\vec{B} + \lambda^2\vec{\nabla} \times \vec{\nabla} \times \vec{B} = \Phi_0\delta^{(2)}(\vec{x} - \vec{x}_p) \tag{11}$$

λ - penetration depth, Φ_0 - flux quantum.

- Isolated vortex solution

$$\vec{v}_S = \frac{\hbar}{m\lambda} K_1\left(\frac{r}{\lambda}\right)\hat{\phi}, \quad K_1\left(\frac{r}{\lambda}\right) \simeq \exp\left(-\frac{r}{\lambda}\right) \quad \text{for} \quad r \gg \lambda. \tag{12}$$

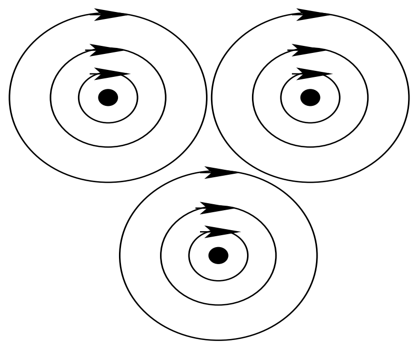
Current is screened beyond the penetration depth !

- Two type of proton superconductivity (in analogy to ordinary superconductors)

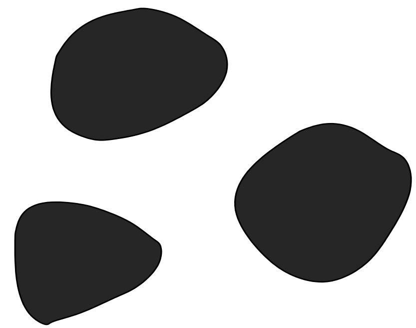
$$\frac{\lambda}{\xi} \leq \frac{1}{\sqrt{2}} \rightarrow \text{type I superconductivity}$$
$$\frac{\lambda}{\xi} \geq \frac{1}{\sqrt{2}} \rightarrow \text{type II superconductivity}$$

- different response to the magnetic field !

Vortex structure for type-II and domain structure for type-I



Vortex structure

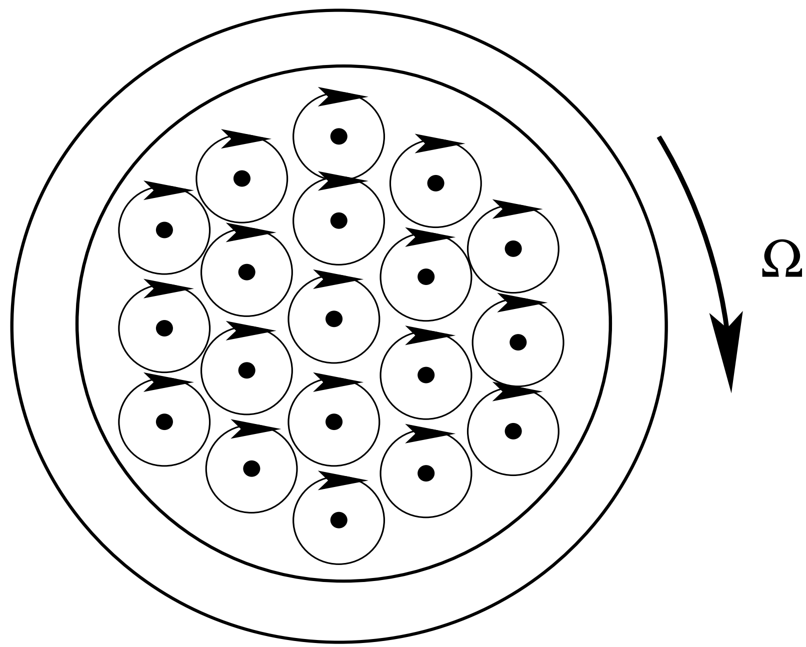


Domain structure

- For type-II the vortex number is $n = B/\Phi_0$.
- For type-I the ratio of domain surfaces $S_S/S_N = B/H_{c2}$.
  The size of domain is not determined and depends on nucleation history

Macroscopic rotation of a superfluid

Macroscopic rotation of a collection of vortices mimics rigid body rotation $\vec{v}_S = \vec{\Omega} \times \vec{r}$ following from $\min[E - \vec{L} \cdot \vec{\Omega}]$



Neutron vortex density (and $\dot{\Omega} \rightarrow$ redistribution of vorticity)

$$n_n = \frac{2m_n\Omega}{\pi\hbar}. \tag{13}$$

Coexisting superfluids and superconductors

- Type-II

$$\frac{\lambda}{\xi_p} > \frac{1}{\sqrt{2}}, \qquad n_p = \frac{B}{\Phi_0} \sim 10^{18}\mathrm{cm}^{-2}, \qquad \frac{n_p}{n_n} \sim 10^{13}. \tag{14}$$

interactions vortex-vortex (generally at an angle) + electron fluid

- Type-I

$$\frac{\lambda}{\xi_p} < \frac{1}{\sqrt{2}}, \qquad n_D \ \text{ unknown}, \ \text{domains carry field } H_{c2}. \tag{15}$$

interactions neutron vortex - normal domain protons + electrons

- But, the physics of at macroscales can be worked out without explicit reference to the mesoscale physics, just in terms of a few phenomenological coefficients.

Minimal superfluid hydrodynamics

- Euler equations (Newtonian dynamics)

$$\rho_S\left[\frac{\partial\vec{v}_S}{\partial t}+(\vec{v}_S\cdot\vec{\nabla})\cdot\vec{v}_S\right] = -\frac{\rho_S}{\rho}\vec{\nabla}p-\rho_S\vec{\nabla}\phi+F, \tag{16}$$

$$\rho_N\left[\frac{\partial\vec{v}_N}{\partial t}+(\vec{v}_N\cdot\vec{\nabla})\cdot\vec{v}_N\right] = -\frac{\rho_N}{\rho}\vec{\nabla}p-\rho_N\vec{\nabla}\phi+\eta_N\Delta\vec{v}_N-F,$$

- Friction force

$$\begin{aligned}\vec{F} &= -\left[\vec{\omega}\times\left(\vec{\nabla}\times\Lambda\vec{\nu}\right)\right]-\beta\left[\vec{\nu}\times\left[\vec{\omega}\times(\vec{v}_N-\vec{v}_S-\vec{\nabla}\times\Lambda\vec{\nu})\right]\right]\\ &- \beta'\left[\vec{\omega}\times(\vec{v}_N-\vec{v}_S-\vec{\nabla}\times\Lambda\vec{\nu})\right]+\beta''\vec{\nu}\cdot\left[\vec{\omega}\cdot(\vec{v}_N-\vec{v}_S-\vec{\nabla}\times\Lambda\vec{\nu})\right],\end{aligned}$$

- Vortex line velocity v_L

$$\frac{\partial\vec{\omega}}{\partial t}=\vec{\nabla}\times(\vec{v}_L\times\vec{\omega}), \tag{17}$$

- Vortex line velocity

$$\vec{v}_L = \vec{v}_S + \vec{\nabla} \times \Lambda\vec{\nu} + \beta'(\vec{v}_N - \vec{v}_S - \vec{\nabla} \times \Lambda\vec{\nu}) \\ +\beta\left[\vec{\omega} \times (\vec{v}_N - \vec{v}_S - \vec{\nabla} \times \Lambda\vec{\nu})\right]. \tag{18}$$

- Force balance equation

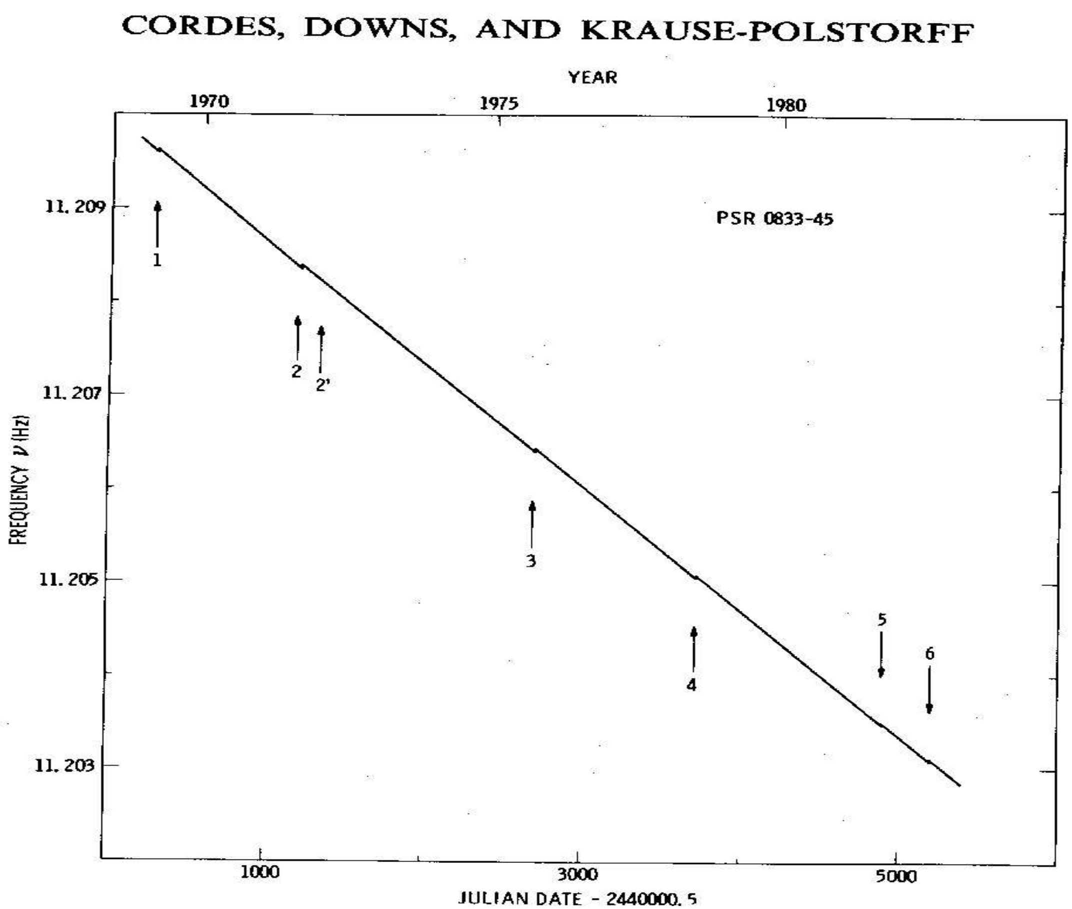
$$\rho_S\left[\left(\vec{v}_S + \vec{\nabla} \times \Lambda\vec{\nu} - \vec{v}_L\right) \times \vec{\omega}\right] \\ -\eta\left(\vec{v}_L - \vec{v}_N\right) + \eta'\left[(\vec{v}_L - \vec{v}_N) \times \vec{\nu}\right] = 0, \tag{19}$$

- Dimensionless drag-to-lift ratios are the key parameters:

$$\zeta = \frac{\eta}{\rho_S\omega} \qquad \zeta' = \frac{\eta'}{\rho_S\omega} \tag{20}$$

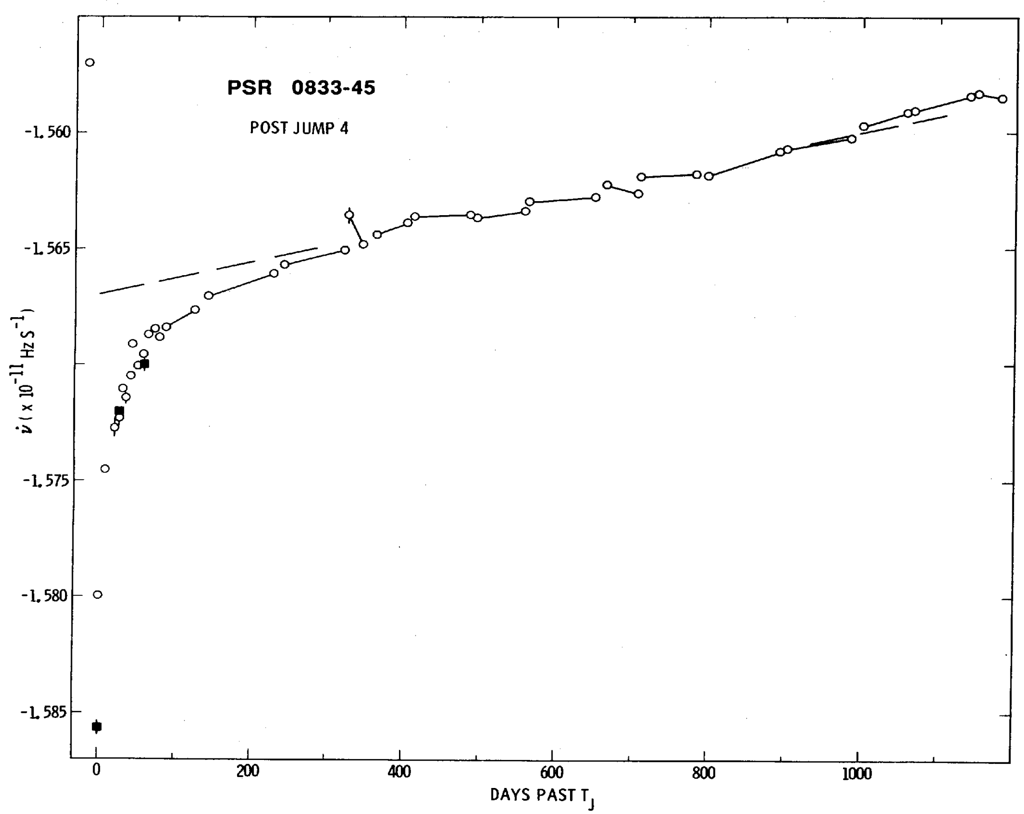
Evidence for S-fluidity: Glitches and post-glitch relaxation

Short time-scale (unresolved) jumps: $\Delta\Omega/\Omega \sim 10^{-5}$ and $\Delta\dot{\Omega}/\dot{\Omega} \sim 10^{-3}$ followed by slow relaxation.



Vortex dynamics theory predicts exponential relaxation which depends on the (density dependent) lift-to-drag ratio ζ, whereby $\tau = (1/2\Omega)\left[\zeta + \zeta^{-1}\right]$ (limits!)

CORDES, DOWNS, AND KRAUSE-POLSTORFF



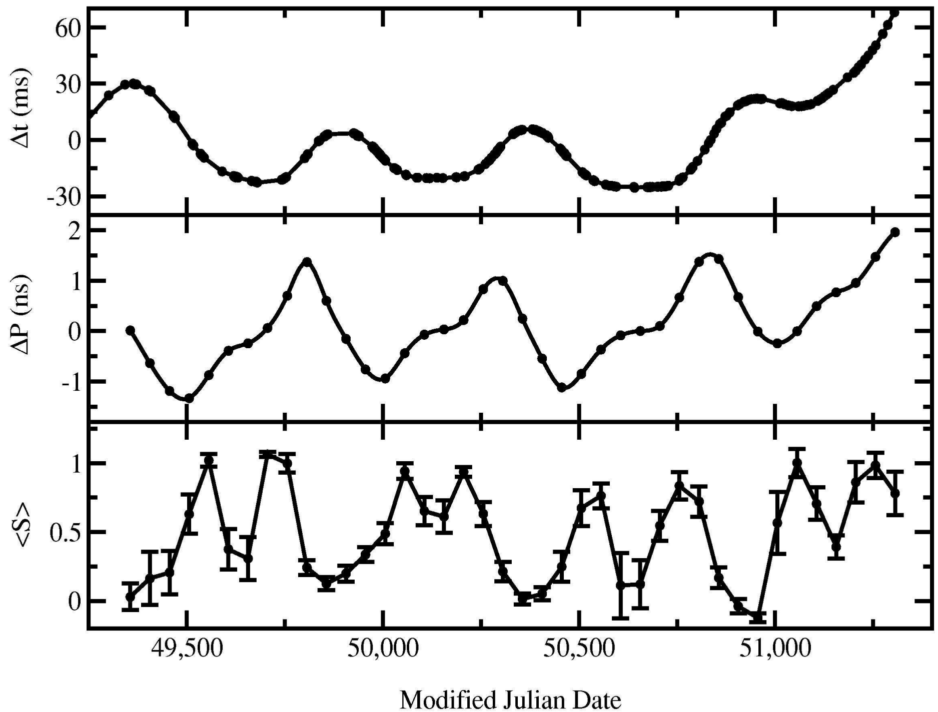
Fits to the data allows to identify relaxation time scales

$$\nu(t) = \nu_0 - \frac{\nu_0}{\tau_0} t - \sum_i \frac{I_{Si}}{I_c} \left(\frac{\nu_0}{\tau_0} \tau_i - \Delta\nu_{Si} \right) \left(1 - e^{-t/\tau_i} \right) \tag{21}$$

$$\dot{\nu}(t) = -\frac{\nu_0}{\tau_0} - \sum_i \frac{I_{Si}}{I_c} \left(\frac{\nu_0}{\tau_0} \tau_i - \Delta\nu_{Si} \right) \frac{e^{-t/\tau_i}}{\tau_i} \tag{22}$$

Unfortunately there is a <u>degeneracy</u> in the interpretation: the strong and weak coupling theories can equally explain the data! But precession lifts this degeneracy!

Precession in PSR B1828-11 (data from I. Stairs et al)



Arrival time residuals, period residuals, and pulse shape parameter

($S = 0$ narrow, $S = 1$ broad). Oscillations with 250, 500 and 1000 days

Classical precessing body

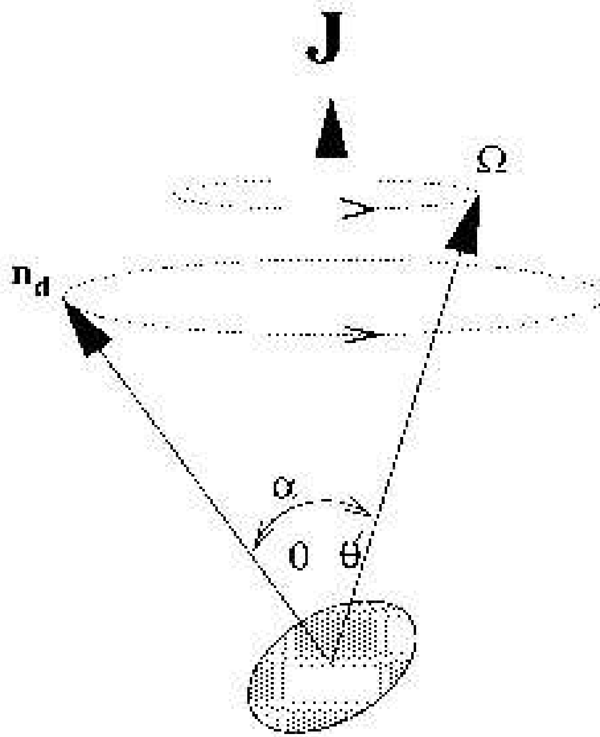


Figure 1. The constant angles in free precession. The symmetry axis n_d and spin axis Ω span a plane containing J. In the inertial frame, n_d and Ω rotate about J at approximately the spin frequency.

How superfluid interiors modifies the precession dynamics?

Precession in a two-component star

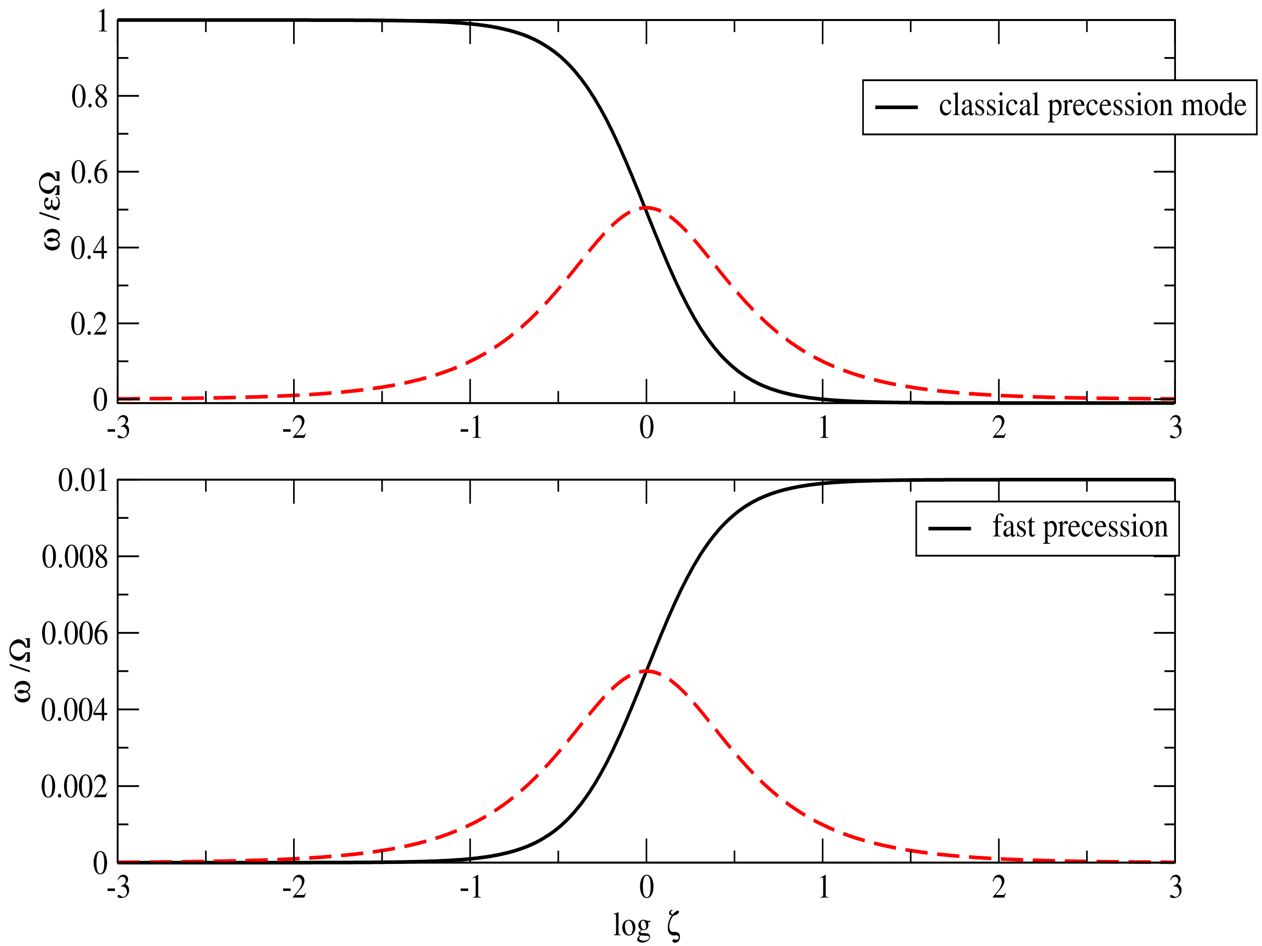
- Divide the star into multiple shells with fixed moment of inertia and the drag-to-lift ratios ζ, ζ'
- Solve the problem for a single shell; add further shells
- Local hydro equations are then integrated over a given shell:

$$
\begin{aligned}
\frac{d(\mathbf{I_{cr}} \cdot \boldsymbol{\Omega}_{cr})}{dt} &= I_s \beta \Omega_s (\boldsymbol{\Omega}_s - \boldsymbol{\Omega}_{cr}) \cdot (\boldsymbol{\delta} + \hat{\boldsymbol{\Omega}}_s \hat{\boldsymbol{\Omega}}_s) + I_s \beta' (\boldsymbol{\Omega}_s \times \boldsymbol{\Omega}_{cr}) \\
&= -\mathbf{N}_\beta - \mathbf{N}_{\beta'} \qquad (23) \\
I_s \frac{d\boldsymbol{\Omega}_s}{dt} &= -I_s \beta \Omega_s (\boldsymbol{\Omega}_s - \boldsymbol{\Omega}_{cr}) \cdot (\boldsymbol{\delta} + \hat{\boldsymbol{\Omega}}_s \hat{\boldsymbol{\Omega}}_s) + I_s \beta' (\boldsymbol{\Omega}_{cr} \times \boldsymbol{\Omega}_s) \\
&= \mathbf{N}_\beta + \mathbf{N}_{\beta'}, \qquad (24)
\end{aligned}
$$

where β and β' are a different set of coeff. related to ζ and ζ'.

model by Sedrakian, Wasserman, and Cordes (ApJ 99)

Small amplitude solutions



Precession modes as a function of the lift-to-drag ratio

- Other factors as multiple shell, external torques, etc do not change this conclusion.
- We see that large ζ exclude precession, observation of the precession discriminates between weak and strong coupling theories

Our conclusions is a no-go theorem for precession:

Precession is impossible if there is a superfluid shell inside the star with drag-to-lift ratio $\zeta \gg 1$

Most of the theories (vortex creep in the crust or vortex cluster friction in the core) predict strong coupling, i. e. no precession.

Alternative: Tkachenko modes

First suggested by Ruderman in 1970 in a short letter to Nature, but no detailed studies in the context of neutron stars:

$$\frac{\partial \mathbf{j}}{\partial t} + (2\boldsymbol{\Omega} \times \mathbf{j}) + \nabla_k \tau_{ik} + \boldsymbol{\sigma} + \vec{\nabla} P + \rho \vec{\nabla} \phi = 0, \tag{25}$$

$$\frac{\partial \mathbf{w}}{\partial t} + (2\boldsymbol{\Omega} \times \mathbf{w}) - \frac{\boldsymbol{\sigma}}{\rho_S} - \mathbf{f} = 0, \tag{26}$$

$$\frac{\partial \mathbf{v}_S}{\partial t} + \left(2\boldsymbol{\Omega} \times \frac{\partial \boldsymbol{\epsilon}}{\partial t}\right) + \frac{\vec{\nabla} P}{\rho} + \vec{\nabla} \phi = 0, \tag{27}$$

the vortex elastic force density defined is

$$\boldsymbol{\sigma} = \mu_S \left[2\vec{\nabla}_\perp \cdot (\vec{\nabla}_\perp \cdot \boldsymbol{\epsilon}) \right] - 2\Omega\lambda \frac{\partial^2 \boldsymbol{\epsilon}}{\partial z^2}, \quad \lambda = (\hbar \rho_S / 8 m_N) \ln (b/a) \tag{28}$$

where $\mu_S = \rho_S \hbar \Omega / 8 m_N$ is the shear modulus vortex lattice.

The Newtonian gravitational potential satisfies the equation

$$\nabla^2\phi = \nabla^2(\phi_S + \phi_N) = 4\pi G(\rho_S + \rho_N), \tag{29}$$

The stress tensor

$$\tau_{ik} = -\eta\left(\nabla_i v_{Nk} + \nabla_k v_{Ni} - \frac{2}{3}\delta_{ik}\vec{\nabla}\cdot\mathbf{v}_N\right), \tag{30}$$

where η is the shear viscosity. The mutual friction force is

$$\mathbf{f} = \beta\rho_S\left[\mathbf{n}\times\left[\boldsymbol{\omega}\times\left(\frac{\partial\boldsymbol{\epsilon}}{\partial t} - \mathbf{v}_N\right)\right]\right] + \beta'\rho_S\left[\boldsymbol{\omega}\times\left(\frac{\partial\boldsymbol{\epsilon}}{\partial t} - \mathbf{v}_N\right)\right], \tag{31}$$

where $\mathbf{n} \equiv \boldsymbol{\omega}/\omega$, $\boldsymbol{\omega}$ is the quantum circulation

Perturbation equations

Transverse modes satisfy

$$\vec{\nabla} \cdot \mathbf{j}_t = \vec{\nabla} \cdot \mathbf{w}_t = 0. \tag{32}$$

The perturbation equations

$$\frac{\partial j_i}{\partial t} + (2\epsilon_{lmn}\Omega_m j_n + \sigma_l + k_m \tau_{lm})P_{il} = 0, \tag{33}$$

$$\frac{\partial w_i}{\partial t} + (2\epsilon_{lmn}\Omega_m w_n - \frac{\sigma_l}{\rho_S} - f_l)P_{il} = 0, \tag{34}$$

$$\frac{\partial v_i}{\partial t} + 2\epsilon_{lmn}\Omega_m \frac{\partial \epsilon_n}{\partial t} P_{il} = 0. \tag{35}$$

where the projector $P_{il} = \delta_{il} - k_i k_l / k^2$, $\mathbf{k}$ is the wave vector.

Secular equation

Perturbations as $j_i(t) \sim j_i\, e^{2\Omega pt}$, leads to $\det\,||K_{ij}|| = 0$ where

$$K_{ij} = \begin{pmatrix} p - \tilde{\eta}\alpha d & (\gamma_S h - 1) & -\tilde{\eta}\gamma_S \alpha d & -\gamma_S\gamma_N h \\ d + \gamma_S g & p - \tilde{\eta} & \gamma_S\gamma_N g & -\tilde{\eta}\gamma_S \\ -\hat{\beta} g & -\hat{\beta}^* h & p + \hat{\beta}(d + \gamma_N g) & -\hat{\beta}^*(1 - \gamma_N h) \\ -\hat{\beta}^* g & \hat{\beta} h & \hat{\beta}^*(d + \gamma_N g) & p + \hat{\beta}(1 - \gamma_N h) \end{pmatrix}. \quad (36)$$

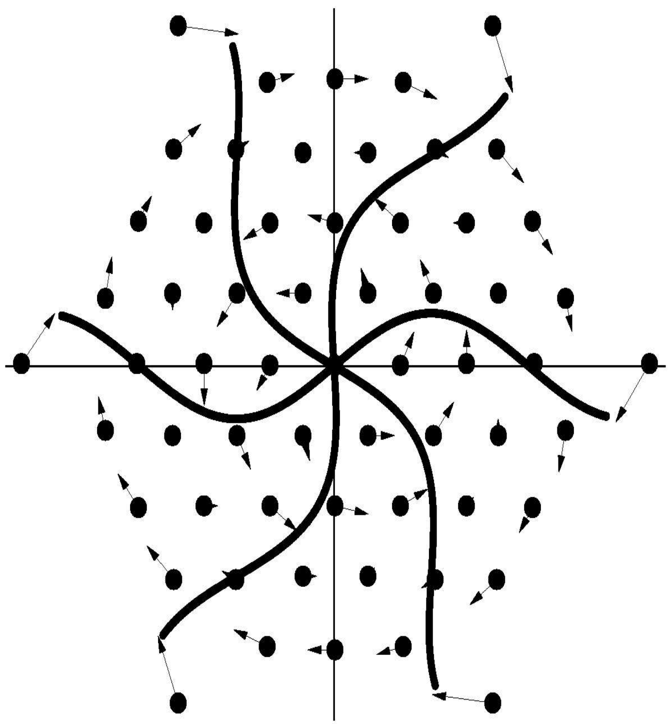
where $\gamma_{N/S} = \rho_{N/S}/\rho$, $d^{1/2} = \cos\ \theta$, $\hat{\beta}^* = 1 - \hat{\beta}'$, $\tilde{\eta} = \eta k2/(2\Omega\rho)$, $\alpha = (4-d)/3$, $\hat{\beta} = \gamma_N^{-1}\beta$, and $\hat{\beta}' = \gamma_N^{-1}\beta'$ and

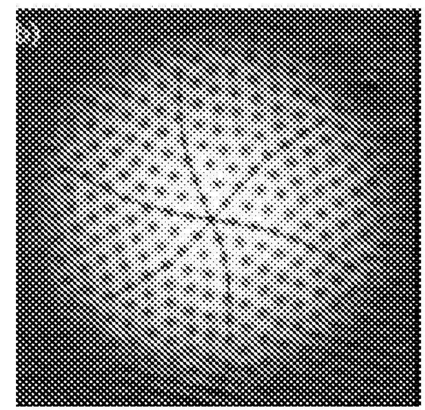
$$g = \frac{k^2}{4\Omega^2 \rho_S}\left[\mu_S - d(\mu_S - 2\Omega\lambda)\right], \quad h = \frac{k^2}{4\Omega^2 \rho_S}\left[\mu_S - d(\mu_S + 2\Omega\lambda)\right]. \quad (37)$$

The (real) eigenfrequencies of these modes in units of 2Ω are

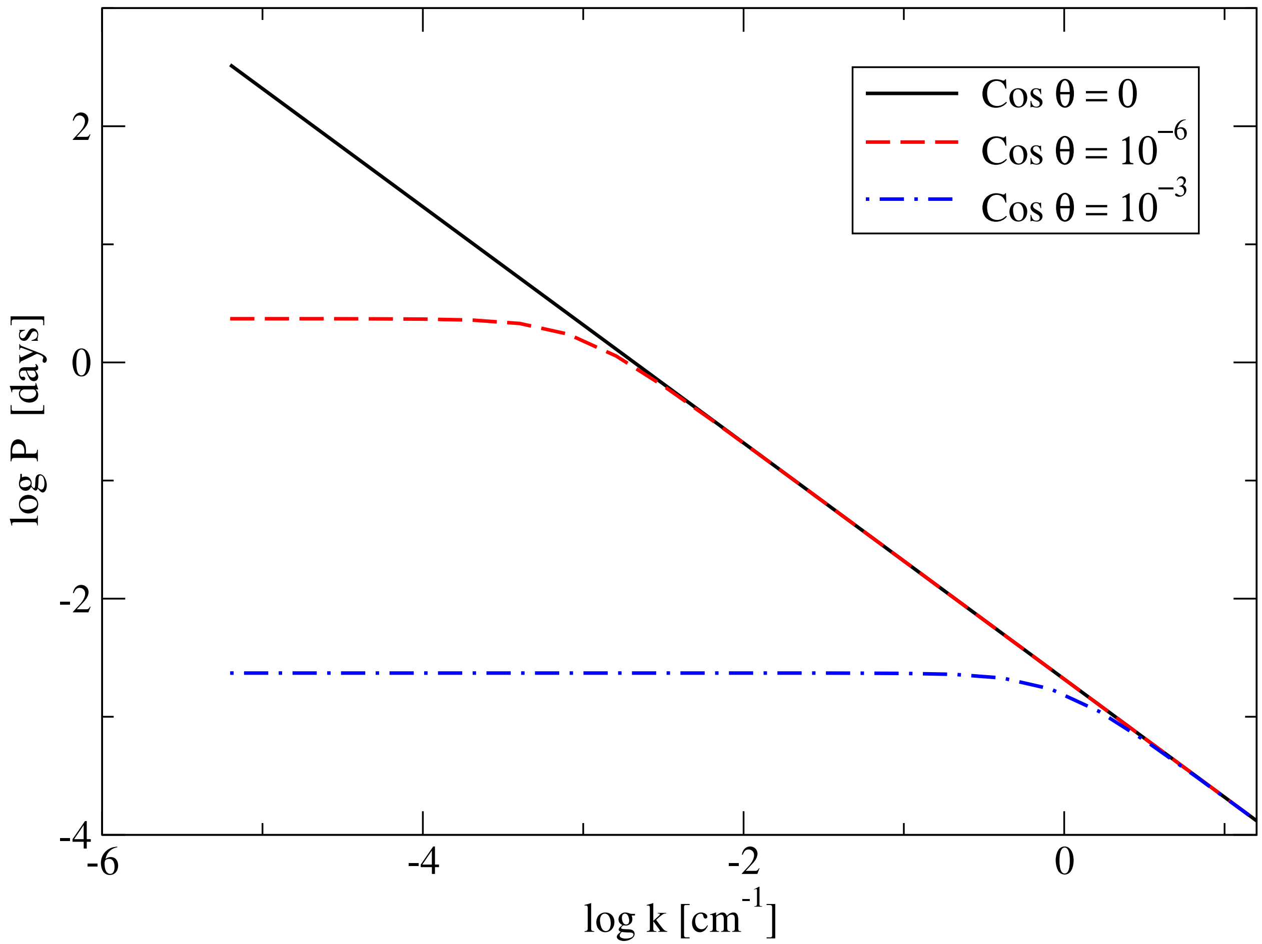
$$p_I \;=\; \pm i\, d^{1/2}, \quad p_T = \pm i\left[(d+g)\,(1-h)\right]^{1/2}, \quad (38)$$

Illustration of the Tkachenko modes

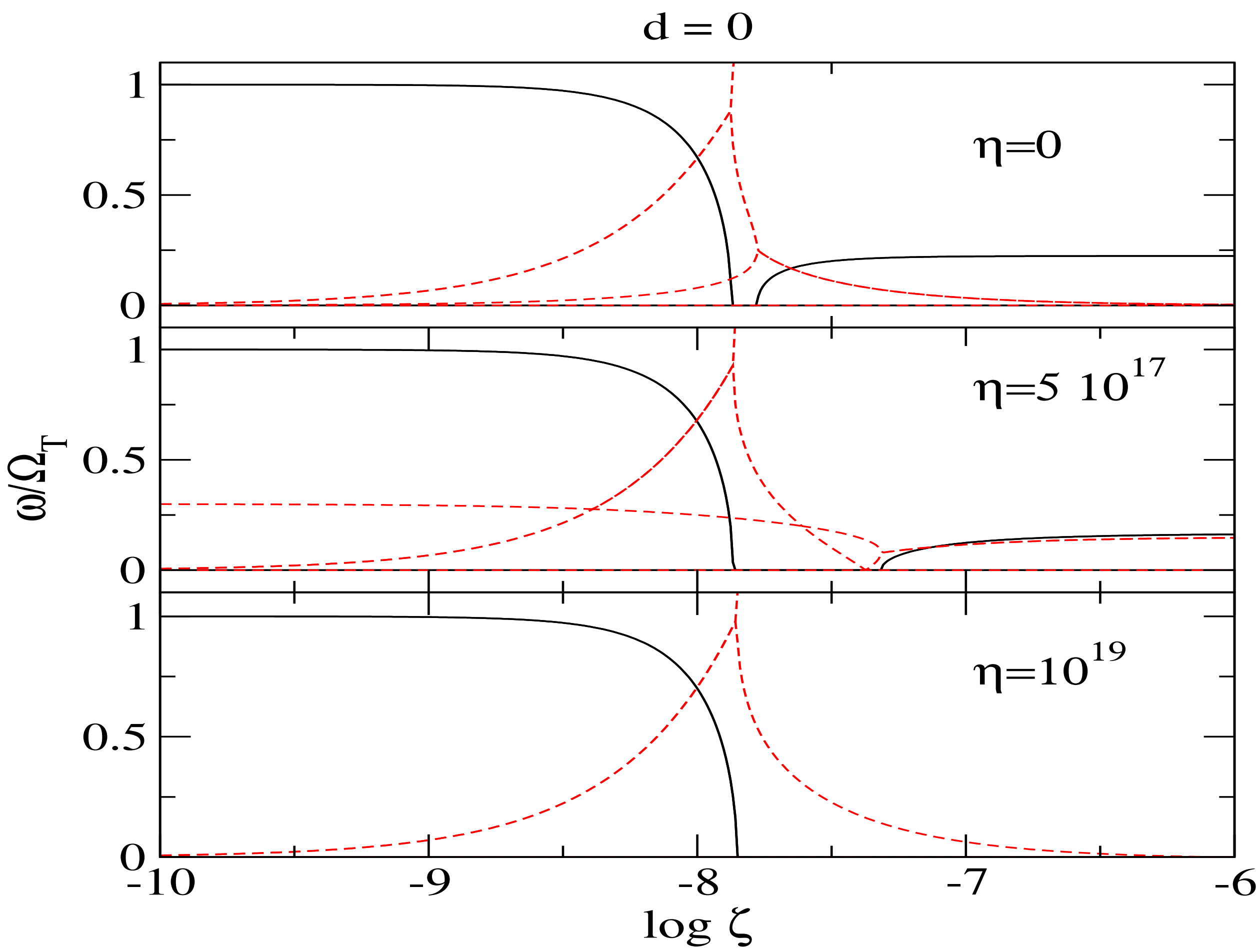




Left panel: theoretical computation, right panel - JILA experiment on BEC



Wave-mode dependence of the Tkachenko modes



Tkachenko modes as a function of lift-to-drag ratio

Conclusion:

- The glitches and post-glitch relaxations can be explained in the strong- and weak-coupling limits. This degeneracy does not allow to pin down the value of the mutual friction coefficient.
- Precession lifts this degeneracy. It could be observed only in the weak-coupling limit. However, many microscopic theories of mutual friction predict strong coupling in the interiors of neutron stars.
- It is unclear why precession is so rare. Interpretation of observation PSR 1828-11 leaves the question why precession is not observed in other systems (mind the timing noise).
- The Tkachenko modes are an interesting alternative for explanation of the quasiperiodic oscillations seen in the timing of pulsars. Much work remains with respect to the understanding of survival of these modes in a complex environment of neutron stars (damping, non-linearities, stratification, etc)