THE VORTICAL MECHANISM OF GENERATION & COLLIMATION OF THE ASTROPHYSICAL JETS

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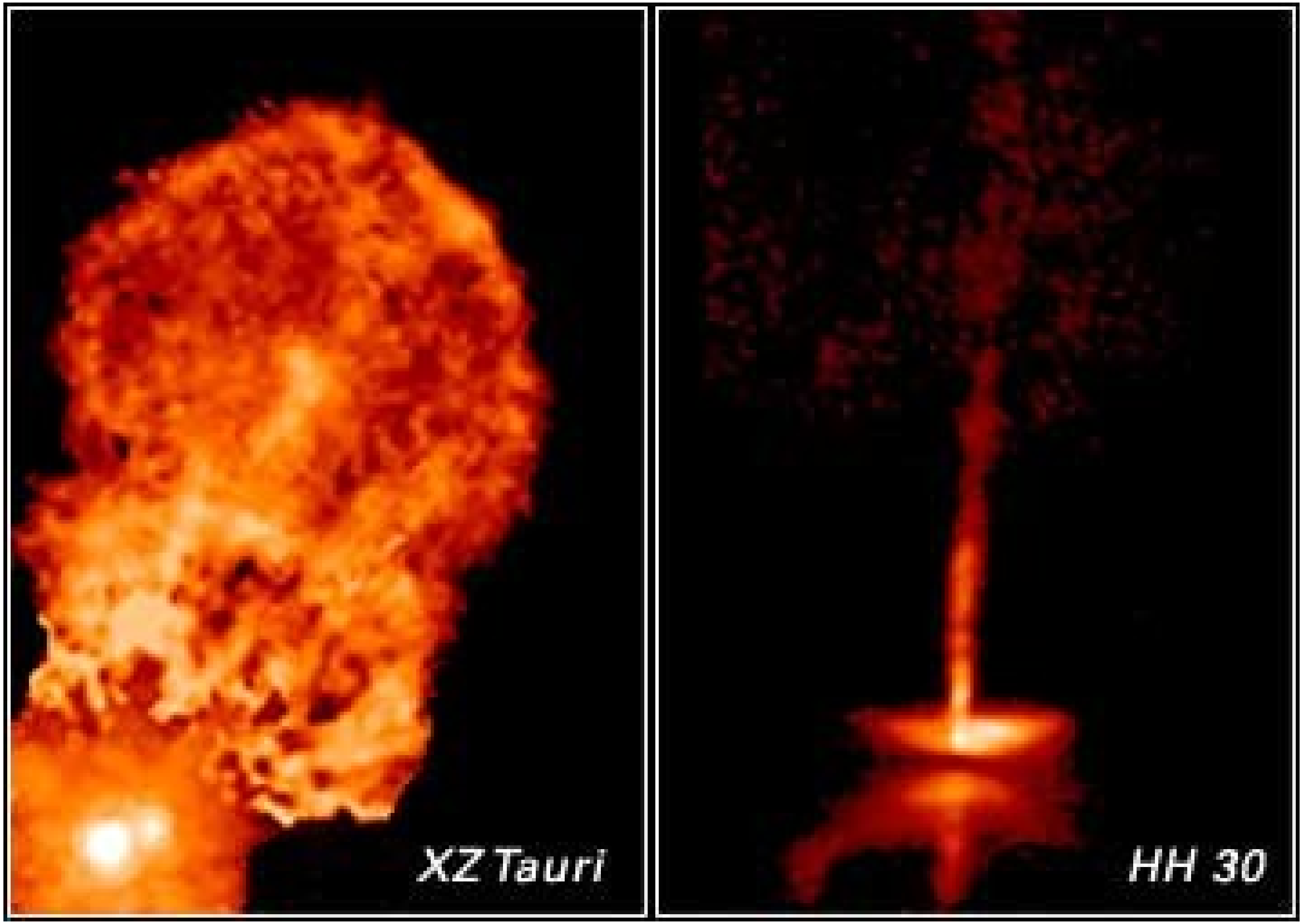
- *Jet eruptive activity*

- *young star formations (YSOs)*
- *neutron stars in a X-ray binary systems,*
- *symbiotic stars,*
- *galactic massive black holes (micro quasars)*
- *gamma ray bursts (GRBs).*
- *nuclear of active galaxies (AGNs)*
- …...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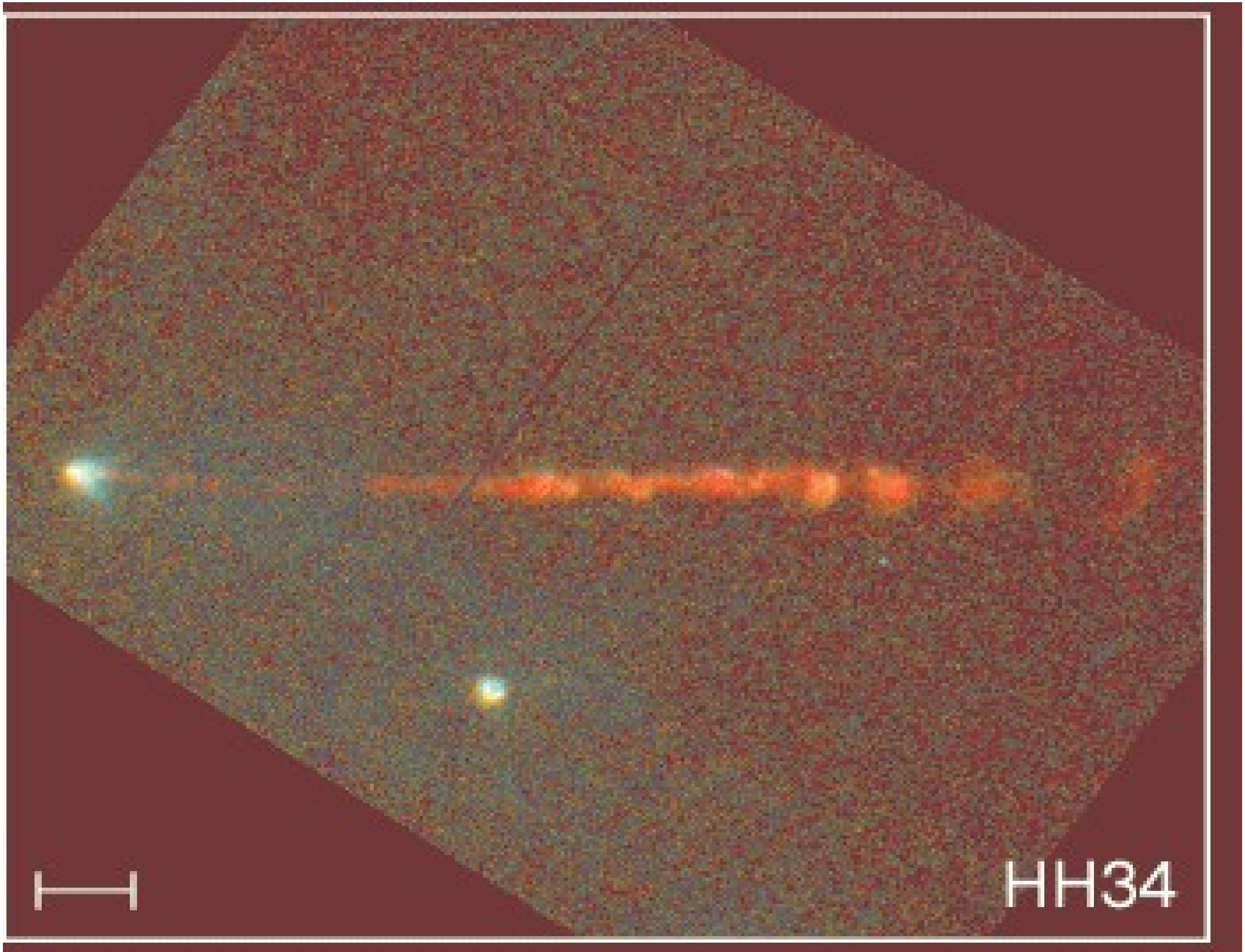
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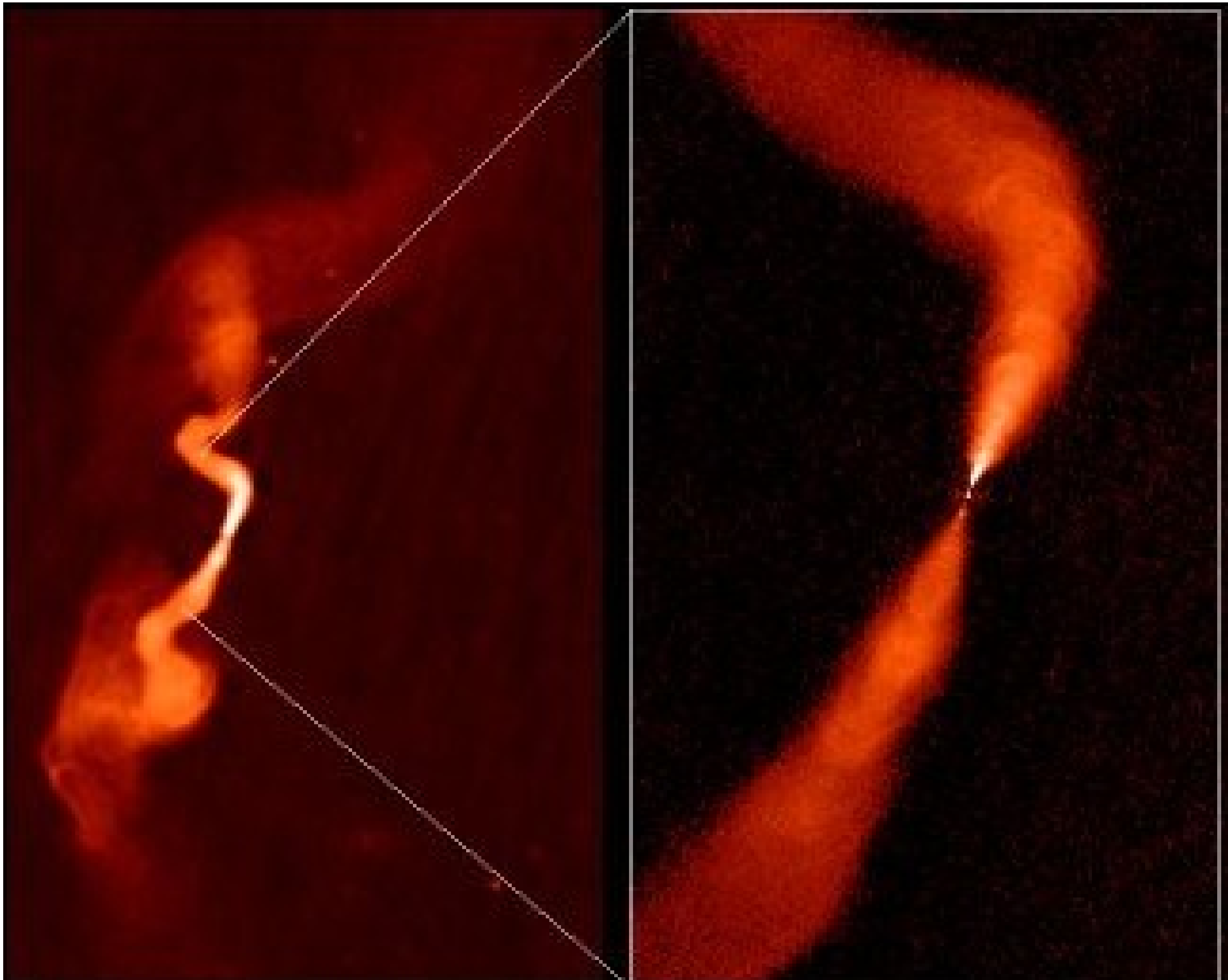

FR Class I source: radio galaxy 3C31

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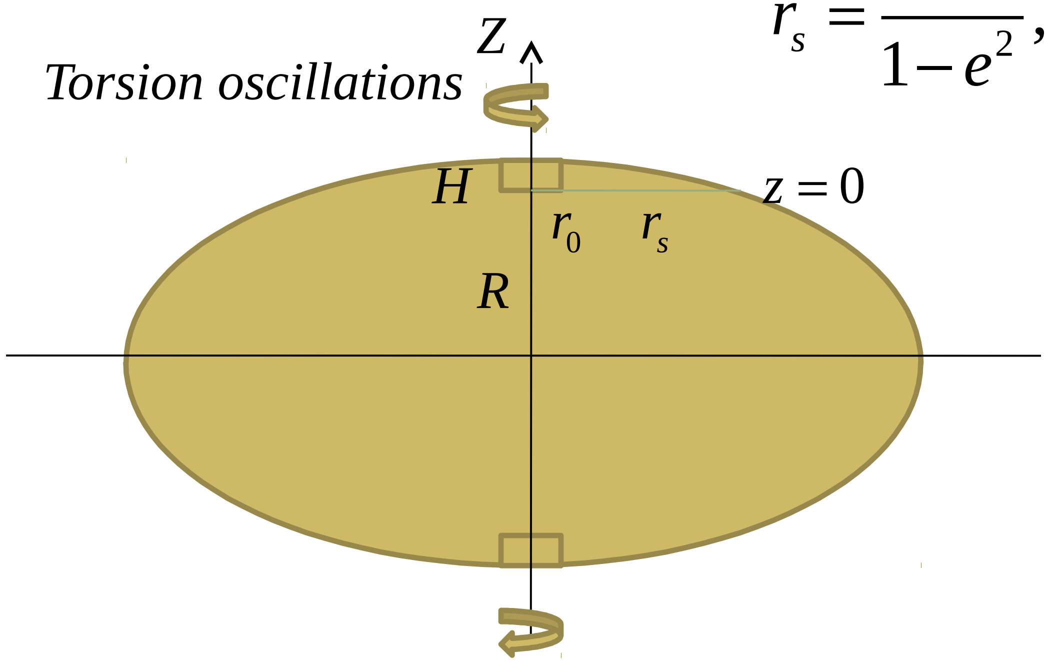
$$r_s = \frac{2RH}{1-e^2}, \quad H = R$$



Equilibrium state of the polar cylindrical region

$$\frac{\partial P^0(r,z)}{\partial r} = -\rho\Omega_0^2(1-e^2)r;$$

$$\frac{\partial P^0(r,z)}{\partial z} = -\rho\Omega_0^2(R-H+z); \quad H = R$$

$$\Omega_0^2 = 2\pi G\rho A; \quad A = \frac{2}{e^2} - \frac{2\sqrt{1-e^2}}{e^3}\arcsin e$$

$$P^0(r,z) = \frac{1}{2}\rho\Omega_0^2(2RH-(1-e^2)r^2-2Rz-z^2)$$

Non linear instability of polar region against the Rankine vortex perturbations

Basic equations

$$P = P^0 + p(r,z,t), \quad V_\varphi = \Omega r + v_\varphi$$

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} - \frac{v_\varphi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + 2\Omega v_\varphi + \nu \frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right),$$

$$\frac{\partial v_\varphi}{\partial t} + v_r\left(\frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r}\right) = -2\Omega v_r + \nu \frac{\partial}{\partial r}\left(\frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r}\right),$$

$$\frac{\partial v_z}{\partial t} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \frac{\partial^2 v_z}{\partial z^2},$$

Vortex type perturbation of the polar regions of protostar

$$t = 0, v_r = v_z = 0$$

Rankin vortex

$$v_\varphi(r) = \begin{cases} \omega_0 r, & r \le r_0 \\ \omega_0 r_0^2 / r, & r > r_0. \end{cases}$$

$$\frac{\partial p}{\partial r} = \rho \frac{v_\varphi^2}{r} + 2\rho\Omega v_\varphi, \quad \frac{\partial v_\varphi}{\partial t} = \nu \frac{\partial}{\partial r}\left(\frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r}\right),$$

$$p = \begin{cases} p_c + \rho(\omega_0^2 + 2\Omega\omega_0)\frac{r^2}{2}, & r \le r_0, \\ -\rho r_0^2\left[\frac{\omega_0^2 r_0^2}{2r^2}\left(1 - \frac{r^2}{r_s^2}\right) + 2\Omega\omega_0 \ln\frac{r_s}{r}\right], & r > r_0, \end{cases}$$

$$p_c = -\rho \frac{r_0^2}{2}\left[\omega_{in}^2 + 2\Omega\omega_{in} + \omega_e^2 + 4\Omega\omega_e \ln\frac{r_s}{r_0}\right].$$

$$v_z = \begin{cases} v_{z0} + \alpha z, & r \le r_0, \\ 0, & r > r_0, \end{cases}$$

Longitudinal flow

$$v_r = -\frac{\alpha}{2}\begin{cases} r, & r \le r_0, \\ \dfrac{r_0^2}{r}, & r > r_0. \end{cases}$$

Radial converging flow

$$\frac{d\omega}{dt} = \begin{cases} \alpha(\omega + \Omega), & r \le r_0, \\ \alpha\Omega, & r > r_0. \end{cases}$$

Acceleration of the trunk

$$\omega(t) = \begin{cases} (\omega_0 + \Omega)e^{\alpha t} - \Omega \approx \omega_{in}, & r \le r_0, \\ \omega_0 + \alpha\Omega t \approx \omega_e, & r > r_0. \end{cases}$$

$$\frac{dE_k}{dt} \approx -4\pi\nu\rho r_0^2\{\frac{3}{2}\alpha^2 + \omega_e^2 + 2\Omega\omega_e\}.$$ *Energy dissipation*

Tangential velocity discontinuity

$$[v_\varphi] = V = [\omega_{in}(t) - \omega_e(t)]r_0 = r_0(\omega_0 + \Omega)(e^{\alpha t} - 1) - \Omega r_0 \alpha t,$$

$$[v_z] \equiv U = v_{z0} + \alpha z.$$

The structure of the cylindrical vortex

$$p(r,z,t) = \begin{cases} p_c - \frac{\rho}{2}[\frac{1}{4}\alpha^2 - \omega_{in}^2 - 2\Omega\omega_{in}]r^2 - \rho[(\dot{v}_{z0} + \alpha v_{z0})z + \alpha^2\frac{z^2}{2}] + C, & r \le r_0, \\ -\frac{\rho}{2}[(\frac{r_0^2}{r^2} - \frac{r_0^2}{r_s^2})(\frac{1}{4}\alpha^2 + \omega_e^2) + 4\Omega\omega_e \ln\frac{r_s}{r}]r_0^2, & r > r_0, \end{cases}$$

Isobaric surfaces

in the $r \le r_0$ *region*

$$z^2(r,t) + 2\left(R + \frac{\dot{v}_{z0} + \alpha v_{z0} - \alpha^2 R}{\Omega_0^2 + \alpha^2}\right) z(r,t) - \frac{\omega_{in}^2 + 2\Omega\omega_{in} - \frac{1}{4}\alpha^2}{\Omega_0^2 + \alpha^2} r^2 -$$

$$- \frac{\Omega_0^2(1-\varepsilon^2)}{\Omega_0^2 + \alpha^2}(r_s^2 - r^2) + \frac{\omega_{in}^2 + 2\Omega\omega_{in} + \omega_e^2 + 4\Omega\omega_e \ln r_s / r_0}{\Omega_0^2 + \alpha^2} r_0^2 - \frac{2C(t)}{\rho(\Omega_0^2 + \alpha^2)} = 0,$$

in the $r > r_0$ *region*

$$z^2 + 2Rz - (1-\varepsilon^2)(r_s^2 - r^2) + \frac{\omega_e^2 + \frac{1}{4}\alpha^2}{\Omega_0^2}\left(\frac{r_0^4}{r^2} - \frac{r_0^4}{r_s^2}\right) + \frac{4\Omega\omega_e}{\Omega_0^2} r_0^2 \ln\frac{r_s}{r} = 0.$$

The unknown function $C(t)$ - *from continuity of isobaric surface at the vortex trunk boundary* $r = r_0$

$$z_{in}(r,t) \approx -\frac{\omega_{in}^2 + 2\Omega_0\omega_{in} - \alpha^2/4 - \Omega_0^2(1-\varepsilon^2)}{2(\dot{v}_{z0} + \alpha v_{z0} + \Omega_0^2 R)}(r_0^2 - r^2) + $$

$$+\frac{1-\varepsilon^2}{2R}(r_s^2 - r_0^2) - \frac{\omega_e^2 + \frac{1}{4}\alpha^2 + 4\Omega_0\omega_e \ln r_s/r_0}{2R\Omega_0^2} r_0^2 .$$

$$z_e(r,t) \approx \frac{1-\varepsilon^2}{2R}(r_s^2 - r^2) - \frac{\omega_e^2 + \frac{1}{4}\alpha^2}{2R\Omega_0^2} r_0^4 \frac{r_s^2 - r^2}{r_s^2 r^2} - \frac{2\Omega_0\omega_e}{R\Omega_0^2} r_0^2 \ln\frac{r_s}{r} .$$

Evolution of the isobaric funnel

$$\dot{v}_{z0} - \alpha(v_{z0} - \alpha R) = 0$$

$$v_{z0}(t) = \alpha R(1 - e^{-\alpha t})$$

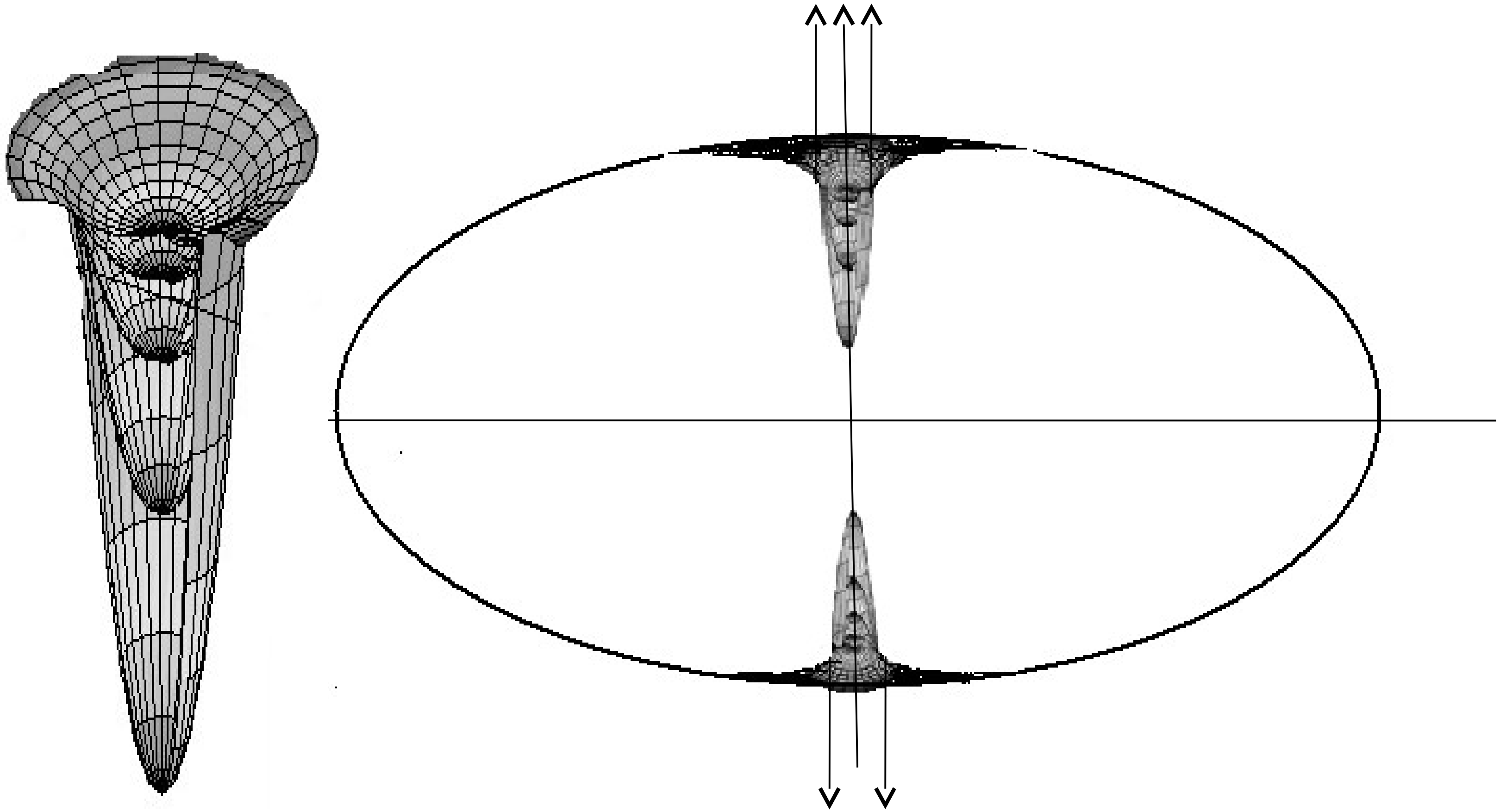
$t = 1/2\alpha$ *funnel deepens linaerely*

$$z(0,t) \sim -\left(\frac{(\Omega + \omega_0)^2 R}{\dot{v}_{z0} + \alpha v_{z0} + \Omega_0^2 R} + \frac{\Omega\omega_0 + 2\Omega^2 \ln r_s / r_0}{\Omega_0^2} \right) \frac{r_0^2}{R} \alpha t$$

$t > 1/2\alpha$ $-$*exponentially*

$$z(0,t) \sim -\frac{(\Omega + \omega_0)^2}{\dot{v}_{z0} + \alpha v_{z0} + \Omega_0^2 R} \frac{r_0^2}{2} e^{2\alpha t}$$

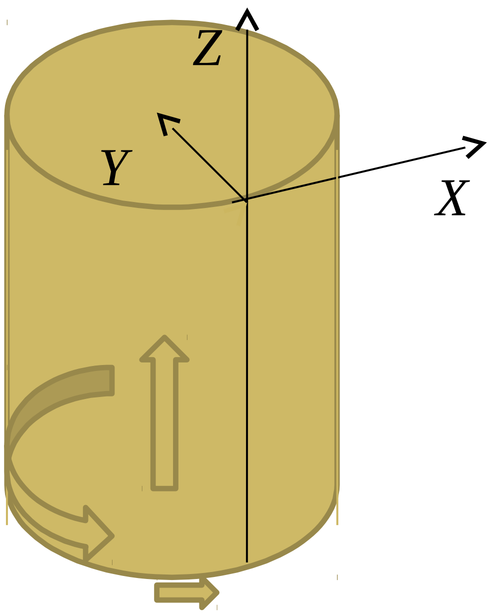
The approximate evolution of an isobaric funnel in the region of a vortex through equal time steps (scale lengths are not maintained).



Instability of the tangential velocity discontinuity at the trunk boundary and the saturation of the vortex.

$$\frac{\partial \mathbf{v}}{\partial t} + V\frac{\partial \mathbf{v}}{\partial x} + U\frac{\partial \mathbf{v}}{\partial z} = -\nabla\frac{P}{\rho} + 2[\mathbf{v}\boldsymbol{\Omega}] + \nu\Delta\mathbf{v}, \quad \nabla\mathbf{v} = 0.$$

$$P/\rho \ \sim \ e^{-k|y|}\, exp\{i(k_x x \ + \ k_z z \ - \ \sigma t)\}.$$



$$\frac{(\sigma - \boldsymbol{kw} + i\nu k^2)^2 - 4\Omega^2}{2\Omega k_x - k(\sigma - \boldsymbol{kw} + i\nu k^2)}(\sigma - \boldsymbol{kw}) =$$

$$= \frac{(\sigma + i\nu k^2)^2 - 4\Omega^2}{2\Omega k_x + k(\sigma + i\nu k^2)}\sigma$$

$Short\ wave: kr_0 \;\; >> 1$

$\sigma_0(k) = Re\sigma = kV/2, \;\; \gamma(k) \; = \; Im\ \sigma; \; = \; [(k^2V^2 + \nu^2 k^4 - 4\Omega kV)^{1/2} - \nu k^2]/2.$

$$k \to \infty, \;\; \gamma_m = V^2/4\nu. \quad \zeta(t) \approx \zeta_0 e^{\gamma_x t} \;:\; \zeta_0 \exp\{\frac{\omega_0^2 r_0^2 \alpha^2}{2\nu} t^3\},$$

In an inviscid fluid $(\nu = 0)$ *surface perturbations with a tangential velocity discontinuity develop during the initial stage as*

$$z(t) \;;\; z_0 \exp\{\tfrac{1}{2} w_0 r_0 a k t^2\}.$$

The maximum growth rate for the perturbations in a layer is known to be attained for wavelengths on the order of its thickness ℓ, i.e., with km≈1lℓ. In the case of surface waves, the layer thickness is ℓ≈2ζ(t), so that km≈1 2ζ(t). So

$$ln \frac{z(t)}{z_0} \;;\; \frac{a w_0 r_0 t^2}{2z(t)},$$

Thus, a turbulent transition layer of thickness 2 ζ(t) develops on the surface of the vortex trunk with an effective turbulent viscosity which, in the initial stage of the development of the instability, can be estimated using the formula

$$n^{*}(t) \; ; \; z^{2}(t)\, g(t) \; ; \; \tfrac{1}{2}\, a v_{0} \left| z(t) \right| t \; ? \;\; n.$$

- *The turbulent perturbations saturate when the rise in the kinetic energy per unit time owing to instability in the tangential velocity discontinuity γρv2 /2, approaches, in order of magnitude, the power of the turbulent énergie dissipation per unit volume, ρv3/ℓ. Where v≈ dζ(t)/dt is the velocity of the turbulent fluctuations, ℓ~ ζ(t) is their characteristic scale length, and γ≈πV/ζ is the maximum growth rate of instability. So, the velocity of the turbulent fluctuations is the same as the discontinuity in the tangential velocity, i.e., v(t)≈V (t).*

- The angular acceleration in the rotation of the vortex trunk ceases when the discontinuity *V(t)* in the tangential velocity approaches the sound speed c_s. *The time* t_s *for this process we obtain giving* $V_m \approx c_s$ & $\gamma_m \approx \pi c_s/\zeta_m$

$$ln \frac{z_m}{z_0} ; \frac{a w_0 r_0 t_s^2}{2 z_m}.$$

The longitudinal long-wave perturbations develop much more slowly than the short-wave perturbations. Thus, saturation of the vortical motion (i.e., termination of the exponential growth in the angular velocity of the trunk and in the pressure drop on its axis) occurs when the discontinuity in the rotational velocity reaches the sound speed. The time *ts for the vortex* t_s *saturate is determined from the equation*

$$e^{\alpha t_s} - \frac{\Omega \alpha t_s}{\omega_0 + \Omega} = \frac{c_s}{(\omega_0 + \Omega) r_0}.$$

Over this time the bottom of the isobaric funnel moves downward by a distance

$$z_s \approx \frac{(c_s + \Omega r_0 \alpha t_s)^2}{2(\dot{v}_{z0} + \alpha v_{z0} + \Omega_0^2 R)}$$

and, the vertical velocity on the plane z=0 increases to

$$v_{z0}(t_s) = \alpha R \left(1 - \frac{(\omega_0 + \Omega) r_0}{c_s + \Omega r_0 \alpha t_s}\right)$$

So the velocity of the jet on the pole of the protostar is

$$v_j \approx \begin{cases} v_{z0} + \alpha(z_s + H), & v_{z0} = const, \\ v_{z0}(t_s) + \alpha(z_s + H), & v_{z0} = \alpha R(1 - e^{-\alpha t}). \end{cases}$$

- *Let’s $R = 10au$ and $e = ¾$. Then $\rho = 3M_{¤}(1-e^2)/2\pi R_0^3 \approx 10^{-11}\ g/sm^3$, and $\Omega \approx 10^{-9}s^{-1}$, $\Omega_0 \approx 2\cdot10^{-9}s^{-1}$.*
- *Let $H = 0.2au$, $r_0 \approx 0.5au$, $v_0 = \omega_0 r_0 \approx 0.3km/s$, $v_r \approx 1km/s$ - on the trunk surface, and a sound velocity of $c_s \approx 50km/s$. Then time of saturation of a vortex: $t_s \approx 2.4\cdot10^8 s \approx 8$ years, cylindrical vortex length $\approx 4au$. Velocity on a protostar surface - $v_j \approx 80\text{-}120\ km/s$.*
- *A free path length of particles: $\zeta_0 \approx m_H / \pi a_H^2 \rho \approx 10^4 sm$, a_H - its Bohr radius. Then $\zeta_m \approx 2\cdot10^7 sm$, turbulent viscosity coefficient $\nu^* = 2\cdot10^{12}\ sm^2/s$, whereas $\nu \approx v_H \zeta_0 /3 \sim 5\cdot10^9\ sm^2/s$.*

$$t_{vortex} \approx 2\frac{10}{[\nu^*]}\,yr, \quad \nu^* - in\ units\ 10^{12}\ sm^2/s$$

Mass loss
$$\frac{dM}{dt} = \pi r_0^2 \rho v_j \approx 10^{-4}\frac{M_{e}}{yr}$$

Summary

- *The appearance of a Rankine vortex in the polar layer of a gravitating body produces a longitudinal flow of matter and a radial converging flow toward the vortex trunk. These flows provide for an exponential growth in the rotational velocity of the trunk and in the pressure drop on its axis.*
- *The power law increases in the angular velocity and in the pressure drop cease and the vortical motion enters a state of saturation when the discontinuity in the rotational velocity at the surface of the trunk reaches the sound speed. During this time the vortical motion extends to ever deeper layers of the protostar.*
- *At the same time the longitudinal velocity along the vortex trunk arises causing mass to flow out through the pole of the protostar as a jet outflow.*

Adiabatic expansion of the naked trunk

Submitted to the journal Astrophysics

- *The vortical mechanism for the generation of astrophysical jets is a unique way of converting gravitational energy of a source into the kinetic energy of an jet outflow.*
- *This mechanism can also provide for the acceleration and collimation of jet flows beyond the confines of a source.*
- *Emerging from the compact body the naked vortex trunk enters a rarefied surrounding medium and begins to expand.*

- *The following scenario for the outflow expansion can be imagined: radial distension and expansion of the surface layers into the rarefied surroundings.*
- *Radial distension converts the jet from a dense, rapidly rotating state into a less dense, more slowly rotating state while conserving its angular momentum.*
- *The matter flows out from the jet surface: initially the layers adjacent to the boundary come into motion, and ever deeper regions away from the boundary are gradually brought into motion. A rarefaction wave develops and propagates into the depth of the jet, creating a "sheath" of no uniform density with a differential rotation.*
- *After this, the rapid expansion processes cease and a pattern consisting of two regions is established in the jet: a core region that is uniform in density and rotates rigidly and a “sheath” region with a no uniform density, rotates differentially, with a converging radial flow of matter.*

Basic equations

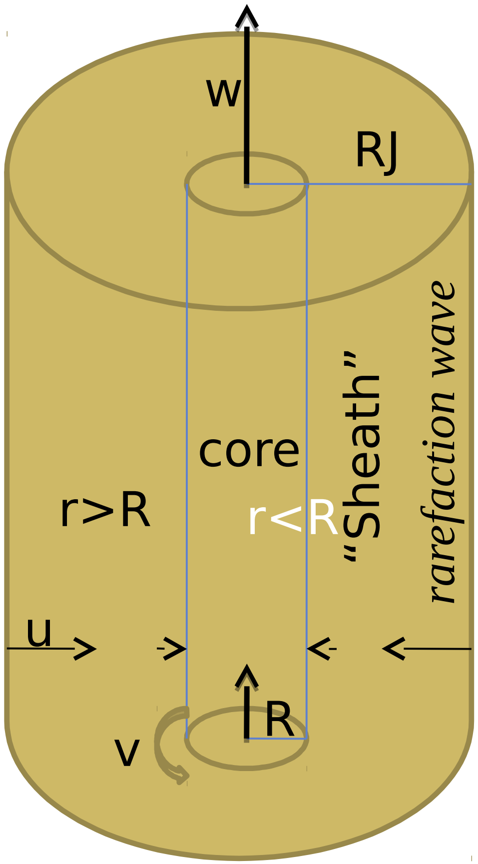
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}-\frac{v^2}{r}=-\frac{\partial h}{\partial r}+\nu_*\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right),$$

$$\frac{\partial v}{\partial t}+u\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right)=\nu_*\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial r}+\frac{v}{r}\right),$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+(V_{j0}+w)\frac{\partial w}{\partial z}=-\frac{\partial h}{\partial z}+\nu_*\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2}\right),$$

$$\frac{\partial \ln \rho}{\partial t}+u\frac{\partial \ln \rho}{\partial r}+\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial w}{\partial z}=0,$$

$$dh=c^2 d\ln\rho,\quad c^2=\gamma p/\rho=(\gamma-1)h,$$



Core region - $r < R$

$$v = \omega_c(t) r,\quad w = \beta z,\quad u = -\tfrac{1}{2}\beta r,$$

$$h = (\omega_c^2 r^2 - u^2)/2 + h_c,$$

$$d\omega_c/dt = \beta\omega_c,\quad \omega_c(t) = \omega_{c0}\exp(\beta t),$$

$$\omega_c(t) = \frac{\omega_{c0}}{1-\omega_{c0} t},\quad \beta = \omega_c$$

Sheath region - $r > R$

$$dh = c^2 d\ln\rho,\quad c^2 = \gamma p/\rho = (\gamma - 1)h,$$

$$\rho_c \omega_c \int_0^R r^2 2\pi r\,dr + \int_R^{R_j} \rho v r 2\pi r\,dr = \frac{1}{2}\pi\rho_0 r_0^3 c_0$$

$$v = \omega_c R,\quad \rho = \rho_c \frac{R}{r},\qquad u = -\frac{v_*}{r},\quad w = -v_* \frac{z}{r^2},\quad R_j = R\left(\frac{c_c}{c_s}\right)^{\frac{2}{\gamma-1}},$$

Pressure drop on axes of the core

$$p_c = \tfrac{1}{2}\rho_c[2h_0(R) + V_r^2 - R^2\omega_c^2(t)], \quad V_r = \tfrac{1}{2}\beta R$$

Energy dissipation:

$$\frac{dE_k}{dt} = -2\pi\nu_*\rho_c R^2\{3\beta^2 + \frac{2}{3}\frac{\nu_*^2}{R^4}\} = const$$

Rotational velocity jump

$$u \equiv [\omega_c(t) - \omega_{c0}]R = \begin{cases} \nu(e^{\beta t} - 1), & \beta = const, \\ \dfrac{\nu\omega_{c0}t}{1 - \omega_{c0}t}, & \beta = \omega_c. \end{cases}$$

$$\frac{p(r,z,t)}{\rho_c} = \begin{cases} -\dfrac{1}{2}[R^2\omega_c^2(t) - V_r^2](1 - \dfrac{r^2}{R^2}) + h_0(R) - 2\dfrac{V_{j0}V_r}{R}z - 2\dfrac{V_r^2}{R^2}z^2, & r \le R, \\ C(t) + \dfrac{\nu_*^2 R}{2\gamma r^3} + \dfrac{\nu^2 R}{\gamma r}\ln r + \dfrac{RV_{j0}\nu_*}{\gamma r^3}z - \dfrac{3R\nu_*^2}{2\gamma r^5}z^2, & r > R. \end{cases}$$

Funnel formation

$$z_{in}^2(r,t) + R\frac{V_{j0}}{V_r} z + \frac{R^2}{4V_r^2}[R^2\omega_c^2(t) - V_r^2](1 - \frac{r^2}{R^2}) + \frac{R^2 h_0}{2V_r^2} + \frac{R^2 P_1}{2V_r^2 \rho_c} = 0$$

$$z_e^2(r,t) - V_{j0}\frac{2r^2}{3\nu_*} z - \frac{r^2}{3} - \frac{2r^4 v^2}{3\nu_*^2}\ln r - \frac{2\gamma r^5 C(t)}{3R\nu_*^2} + \frac{2\gamma r^5 P_1}{3R\nu_*^2 \rho_c} = 0,$$

$$z(0,t) = -\frac{R}{2V_{j0}V_r}\left(\frac{v^2}{2} e^{2\beta t} - \frac{V_r^2}{2} + h_0(R) + \frac{P_1}{\rho_c}\right)$$

$$\dot{z}(0,t) = -\frac{v^2}{V_{j0}} e^{2\beta t}.$$

Saturation of the vortex

Instability of tangential velocity jump on the compressible core boundary

$$\left(\frac{1}{(\omega-uk)^2(c_s^2-i\nu_e\omega)}-\frac{1}{\omega^2(c_s^2-i\nu_e(\omega-uk))}\right)\frac{1}{(\omega-uk)^2(c_s^2-i\nu_e\omega)}+$$

$$+\frac{1}{\omega^2(c_s^2-i\nu_e(\omega-uk))}-\frac{1}{k^2(c_s^2-i\nu_e\omega)(c_s^2-i\nu_e(\omega-uk))}]=0.$$

$$\omega_{1,2}=\omega_1^0-i\frac{c_s^2}{\nu_e}\left(1\pm\sqrt{1-\frac{\nu_e^2}{c_s^4}\omega_1^{02}}\right),\qquad -\ \textit{no instability}$$

$$\omega=\frac{a}{2}\pm\frac{1}{2}\left(a^2-\frac{2b}{3}+A\right)^{1/2}\mp\frac{1}{2}\sqrt{2\left(a^2-\frac{2b}{3}\right)-A\pm\frac{2(a^3-ab-c)}{\left(a^2-\frac{2b}{3}+A\right)^{1/2}}}$$

$$A=\frac{2^{2/3}(b^2+6ac-12s)+(D+\sqrt{G})^{2/3}}{3\cdot 2^{1/3}(D+\sqrt{G})^{1/3}},$$

$$D=-[4c_s^4(4c_s^2-21u^2)+u^2(9k^2\nu_e^2-2u^2)+12c_s^2u^2(u^2-6k^2\nu_e^2)]k^6,$$

$$G=[16c_s^4(8c_s^2-u^2)(c_s^2+u^2)^2-8c_s^2k^2\nu_e^2(8c_s^6+60c_s^4u^2-3c_s^2u^4-u^6)+$$
$$+k^4u^2\nu_e^4(176c_s^4-32c_s^2u^2-u^4)+4k^6u^4\nu_e^6]27u^2k^{12},$$

$$a=(u-i\nu k)k,\ b=-(2c_s^2-u^2+3i\nu_e uk)k^2,\ c=(2c_s^2+i\nu_e uk)uk^3,\ s=c_s^2u^2k^4.$$

$$k\to\infty,\quad \gamma_\infty\approx\frac{u^2+4\gamma_1^2}{4\nu_e}=\frac{c_s\sqrt{u^2+c_s^2}-c_s^2}{\nu_e}$$

Saturation time & max size of the turbulent transition layer

$$t_{\max}\approx\frac{1}{\beta}ln[\frac{c_s}{\omega_{c0}R}],\quad ln\frac{\zeta_{\max}}{\zeta_0}\approx\frac{\beta\omega_{c0}Rt_{\max}^2}{4\zeta_{\max}}.$$

Turbulent viscosity coefficient $\nu_{\max}^{*}\approx\frac{1}{4}\beta\omega_{c0}R\zeta_{\max}t_{\max}$