Vacuum Quantum Effects in Higher-Dimensional Cosmological Models

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- Many of high energy theories of fundamental physics are formulated in higher-dimensional spacetimes

- Idea of extra dimensions has been extensively used in supergravity and superstring theories

Why extra dimensions?

- Unification of interactions
- Extra dimensions offer new possibilities for breaking gauge symmetries
- Topological mass generation
- Hierarchy problem
- Why D=4?

History

- 1914, Nordstrom proposed a 5D vector theory to simultaneously describe electromagnetism and a scalar version of gravity
- 1919, Kaluza noticed that the 5D generalization of Einstein theory can simultaneously describe gravitational and electromagnetic interactions
- 1924, the role of gauge invariance and the physical meaning of the compactification of extra dimensions was elucidated by

History (continued)

- Late 1970's, Higher-dimensional theories were reintroduced in physics to exploit the special properties that supergravity and superstring theories possess for particular values of spacetime dimensions
- More recently it was realized [Arkani-Hamed et al 1998, Randall & Sundrum 1999] that extra dimensions with a fundamental scale of order TeV^{-1} could address the M_W – M_{Pl} hierarchy problem and therefore have direct implications for collider experiments

Types of models

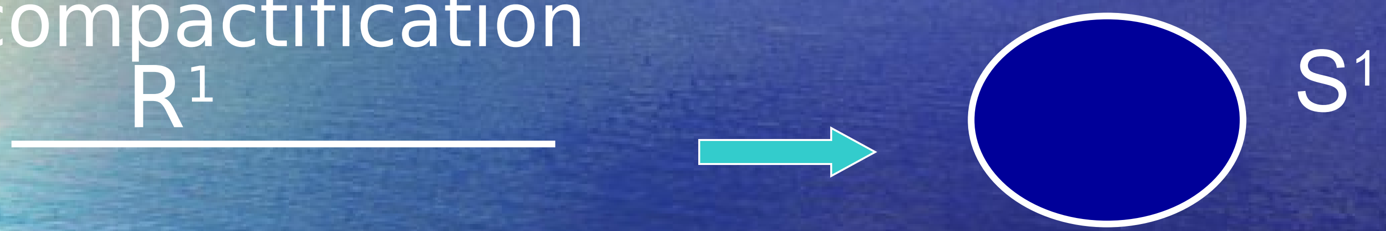
- Kaluza-Klein type models: Extra dimensions are accessible for all fields
- Brane-world models: Part of fields is localized on a hypersurface (brane). The localization can be realized in field theory, but it is most natural in the setting of string theory. Gravity extends to all dimensions
  - The space is factorized into R^4xM and 4D part of the metric does not depend on extra coordinates
  - 4D part of the metric depends on extra coordinates (warped extra dimensions)

Quantum effects in cosmology

- Particle creation
- Vacuum polarization
- Generation of cosmological constant
- Isotropization of the cosmological expansion
- Generation of cosmological inhomogene-ities by quantum fluctuations

Boundary conditions in models with compactified dimensions

Simplest example is the toroidal compactification



Topologically inequivalent field configurations

Untwisted field → periodic boundary conditions

$$\varphi(x+L)=\varphi(x)$$

Twisted field → antiperiodic boundary conditions

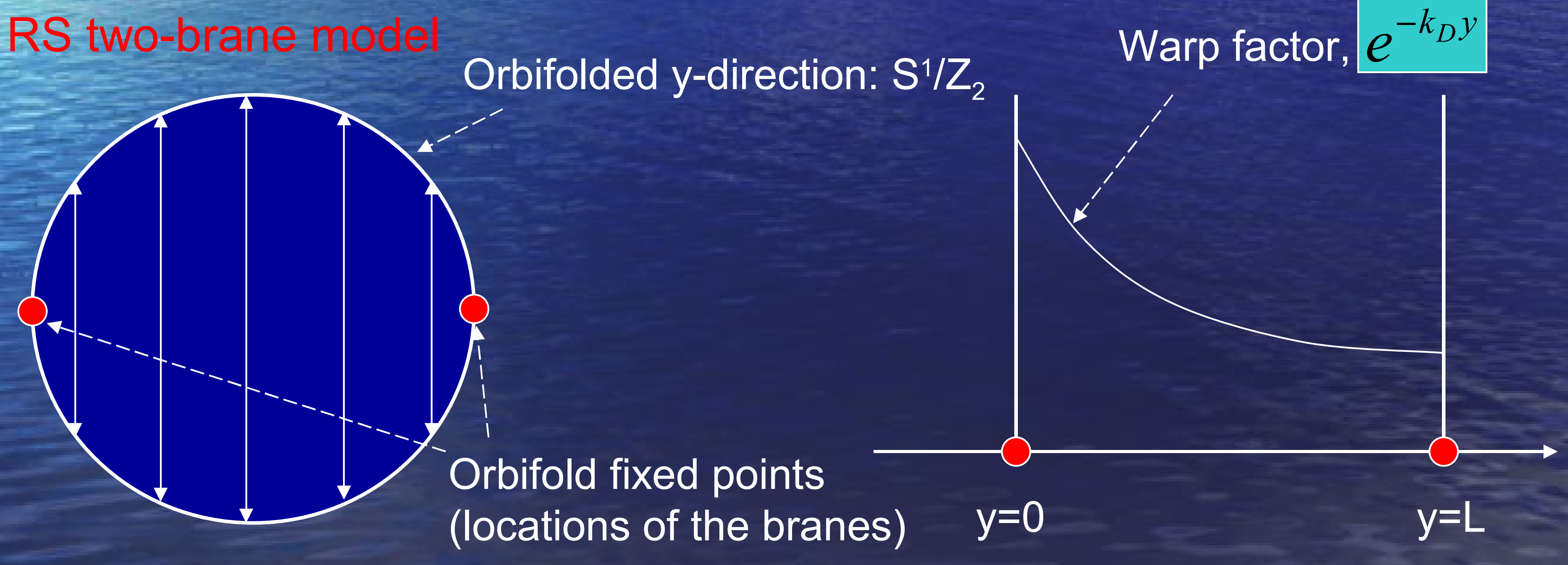
$$\varphi(x+L)=-\varphi(x)$$

Boundary conditions in brane-world models

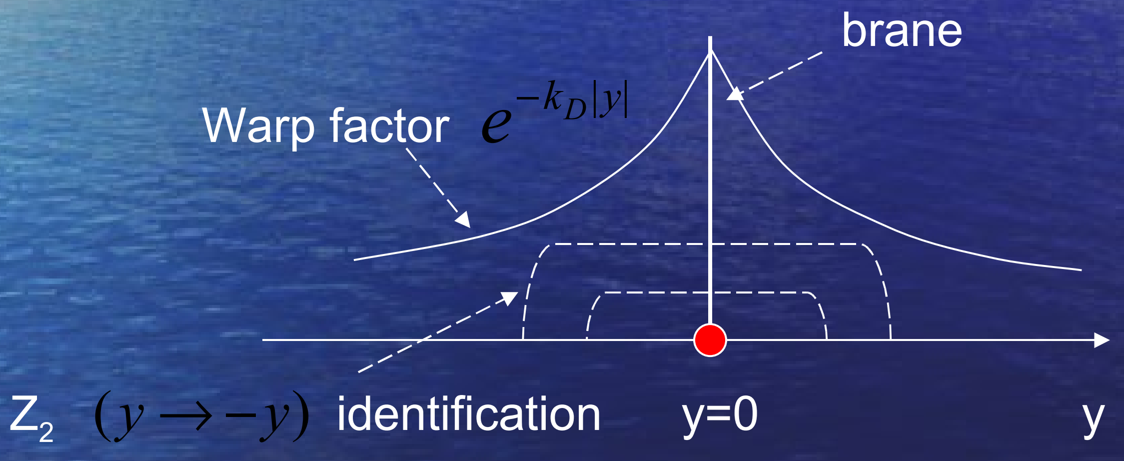
Randall-Sundrum 2-brane model

Bulk geometry: 5-dimensional Anti-de Sitter (AdS) spacetime

$$ds^2 = e^{-2k_D y}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$



Randall-Sundrum 1-brane model



Boundary conditions in RS braneworld

- Boundary conditions for the field are derived integrating the field equation about $y = 0$ and $y = L$
- For the untwisted scalar field ($\varphi(y) = \varphi(-y)$) mixed boundary conditions are obtained with

$$\partial_y \varphi - (c_1/2 + 4Dk\zeta)\varphi = 0, \quad y = a$$
$$\partial_y \varphi - (-c_2/2 + 4Dk\zeta)\varphi = 0, \quad y = b$$

- For the twisted scalar field ($\varphi(y) = -\varphi(-y)$) Dirichlet boundary conditions ($\tilde{B}_a = \tilde{B}_b = 0$) are obtained

In both types of higher-dimensional models the fields propagating in the bulk are subject to boundary conditions

- In Quantum Field Theory the imposition of boundary conditions on the field operator leads to the change of the spectrum for vacuum (zero-point) fluctuations
- As a result the vacuum expectation values of physical observables are changed

Casimir effect

H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948)

- student of Ehrenfest, worked with Pauli and Bohr

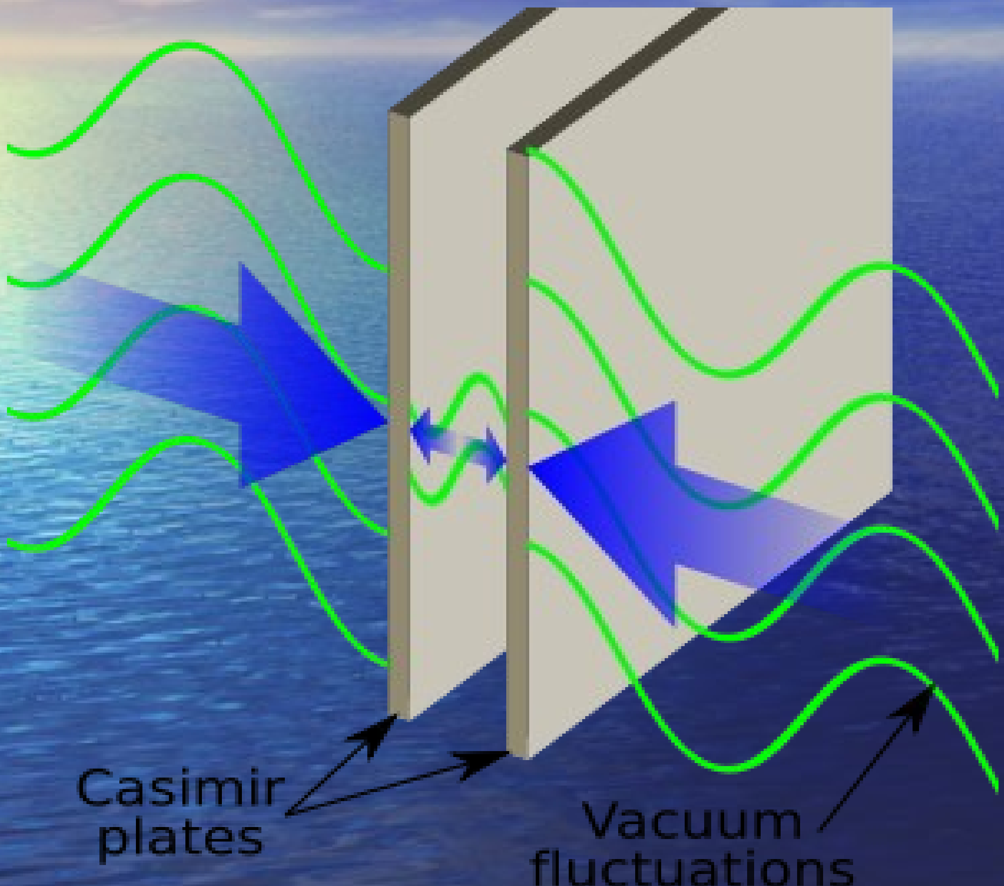
- *Force between cavity walls*

$$F = -\frac{\pi^2 \hbar c}{240} \frac{A}{d^4}$$

Casimir, 1948

Hendrik Brugt Gerhard Casimir, 1909 - 2000

Casimir configuration



$$\frac{F_C}{S} = -\frac{\partial}{\partial a}\frac{E_C}{S} = -\frac{c\pi^{٢}}{٢٤٠a^{٤}}$$

Quantization in theories with boundary conditions

- Complete set of orthonormal solutions to the classical field equations satisfying the boundary conditions

$$\left\{ \psi_\alpha , \psi_\alpha^* \right\}$$

- Expansion of the field operator

$$\hat{\psi} = \sum_\alpha \left(\hat{a}_\alpha \psi_\alpha + \hat{a}_\alpha^+ \psi_\alpha^* \right)$$

- Hamilton and particle number operators

$$\hat{H} = \sum_\alpha \omega_\alpha \left(\hat{N}_\alpha + 1/2 \right), \qquad \hat{N}_\alpha = \hat{a}_\alpha^+ \hat{a}_\alpha$$

- Fock space

vacuum state

$$\hat{a}_\alpha |0\rangle = 0 \qquad \hat{a}_\alpha^+ |0\rangle = |1_\alpha\rangle \qquad \hat{N}_\alpha |0\rangle = 0$$

Vacuum in QFT

- **Vacuum** = state of a quantum field with zero number of quanta

$$\hat{N}|0\rangle = 0$$

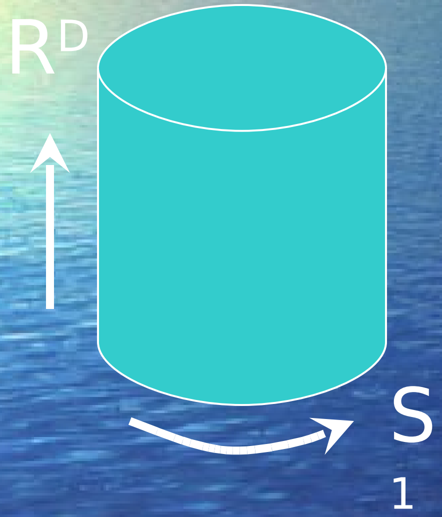
- Vacuum is an eigenstate for the Hamilton operator

$$\hat{H}|0\rangle = \frac{1}{2}\sum_{\alpha}\omega_{\alpha}|0\rangle$$

- Commutator $[\hat{\psi},\hat{H}] \neq 0 \Rightarrow$ In the vacuum state the field fluctuates: Vacuum or zero-point fluctuations ⇒ Nontrivial properties of the vacuum
- Vacuum properties depend on the zero-point fluctuations spectrum
- External fields or boundary conditions imposed on a quantum field change the spectrum of vacuum fluctuations
- Vacuum expectation values of local physical observables are changed (vacuum polarization)

Vacuum quantum effects in models with non-trivial topology

Simple example: Flat model with topology $R^D x S^1$



Vacuum energy density for an untwisted massive scalar field

$$\rho = \frac{1}{2}\int \frac{d^D k}{(2\pi)^D} \sum_{n=-\infty}^{+\infty} \left[(2\pi n / a)^2 + k^2 + m^2 \right]^{1/2}$$

For twisted scalar filed

$$\rho = \frac{1}{2}\int \frac{d^D k}{(2\pi)^D} \sum_{n=-\infty}^{+\infty} \left[(\pi(2n+1) / a)^2 + k^2 + m^2 \right]^{1/2}$$

Renormalized vacuum energy and stresses in topology $R^D x S^1$

- Vacuum energy density

$$\rho = -\frac{2m^{D+2}}{(2\pi)^{D/2+2}}\sum_{n=1}^{\infty}\frac{K_{D/2+1}(nam)}{(nam)^{D/2+1}}$$

- Vacuum stresses in uncompactified subspace

$$P = -\rho$$

- Vacuum stresses in compactified subspace

$$P_{D+1} = -\rho - \frac{2m^{D+2}}{(2\pi)^{D/2+2}}\sum_{n=1}^{\infty}\frac{K_{D/2+2}(nam)}{(nam)^{D/2}}$$

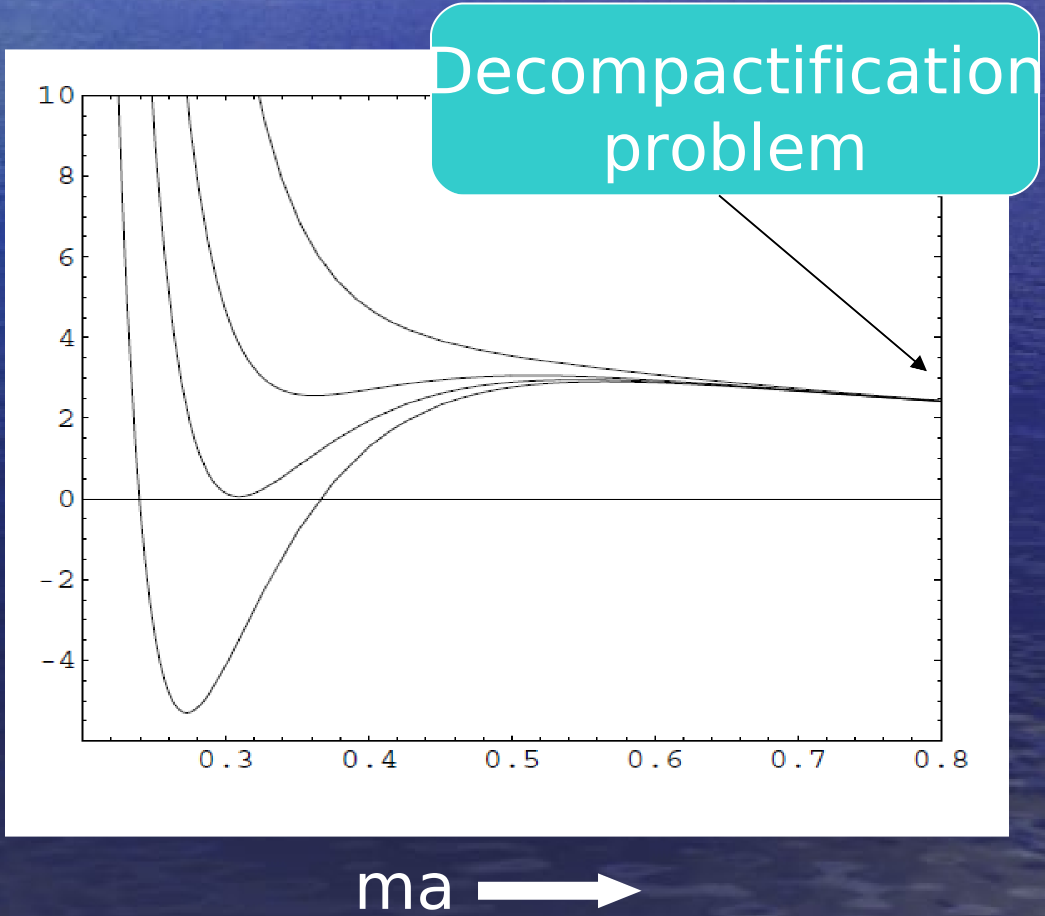
$$(-1)^n$$

For twisted scalar

Effective potential for the size of compactified dimension

Vacuum densities appear in the cosmological equations as an effective potential for the scale factor

Effective potential for a combination of untwisted and twisted scalar fields with masses m and m_t



Graphs are plotted for different values of m/m_t

Quantum vacuum effects in toroidally compactified de Sitter space-time

- A. A. Saharian, M. R. Setare, *Phys. Lett.* B659, 367 (2008).
- S. Bellucci, A. A. Saharian, *Phys. Rev.* D77, 124010 (2008).
- A. A. Saharian, *Class. Quantum Grav.* 25, 165012 (2008).
- E. R. Bezerra de Mello, A. A. Saharian, arXiv:0808.0614.

Why de Sitter space-time?

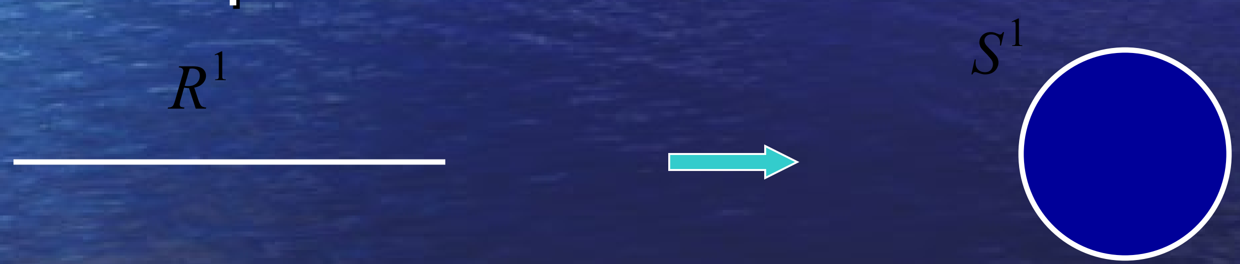
- In most inflationary models an approximately dS spacetime is employed to solve a number of problems in standard cosmology
- At the present epoch the universe is accelerating and can be well approximated by a world with a positive cosmological constant
- Due to the high symmetry numerous physical problems are exactly solvable on dS background and a better understanding of physical effects in this bulk could serve as a handle to deal with more complicated

Background geometry

$(D+1)$-dimensional De Sitter spacetime with spatial topology $\mathrm{R}^p \times (\mathrm{S}^1)^q$ and the line element

$$ds^2 = dt^2 - e^{2t/\alpha} \sum_{i=1}^{D} (dz^i)^2, \quad 0 \leqslant z^l \leqslant L_l \;, l = p+1, \ldots, D$$

Toroidal compactification



Local geometry is the same, the global properties of the space are not

Vacuum energy-momentum tensor

In dS spacetime with topology $R^p \times (S^1)^q$ the vacuum expectation value of the energy-momentum tensor is presented in the form

$$\langle T_i^k \rangle_{p,q} = \langle T_i^k \rangle_{\mathrm{dS}} + \langle T_i^k \rangle_{\mathrm{c}}$$

EMT for uncompactified dS space-time

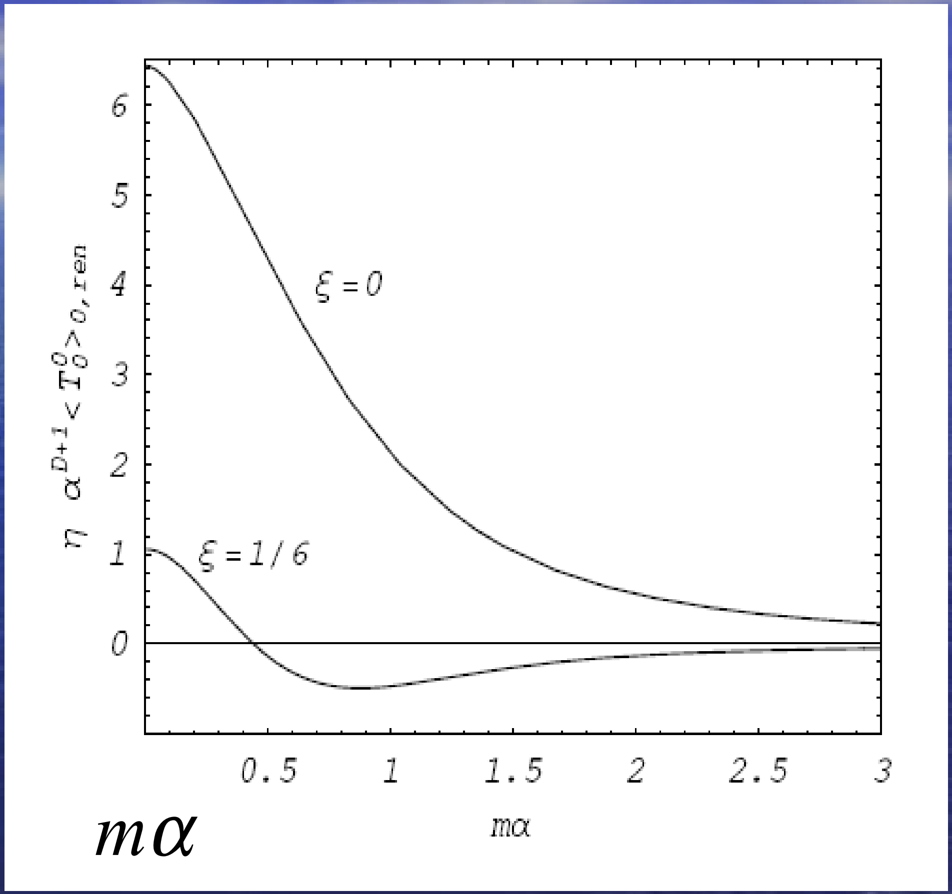
topological part

$\langle T_i^k \rangle_{\mathrm{dS}}$ ⇒ gravitational source of the cosmological constant type

$\langle T_i^k \rangle_{\mathrm{c}}$ ⇒ is time dependent and breaks the dS invariance

Vacuum energy density in uncompactified dS spacetime

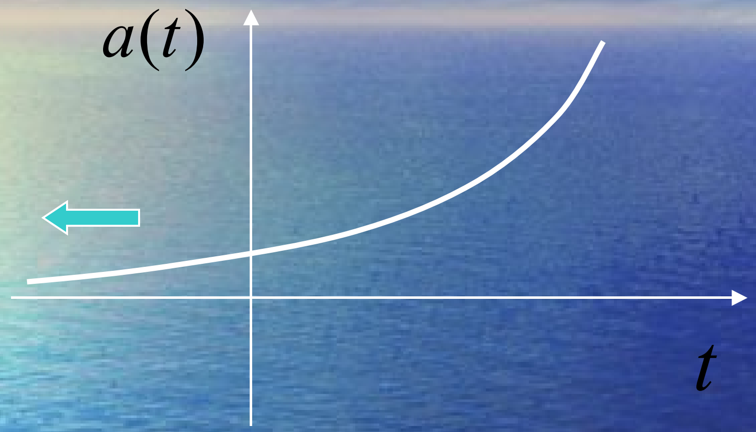
$$\eta\, \alpha^{D+1} \left\langle T_0^0 \right\rangle_{\mathrm{dS}}$$



Renormalized vacuum energy density in uncompactified dS spacetime for minimally and conformally coupled scalar fields in D = 3. The scaling coefficient $\eta = 10^3 (10^4)$ for minimally (conformally) coupled scalar fields.

Properties of the topological part

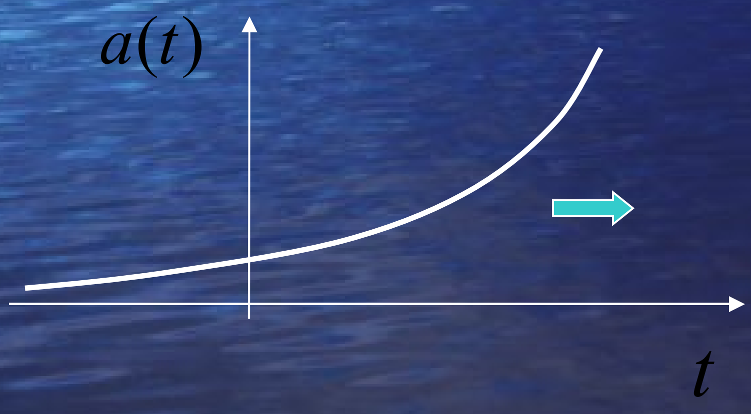
Early stages of the cosmological evolution, $t \to -\infty$



$$\left\langle T_i^k \right\rangle_c \sim [a(t)]^{-D-1}, \quad a(t) = e^{t/\alpha}$$

Topological part dominates

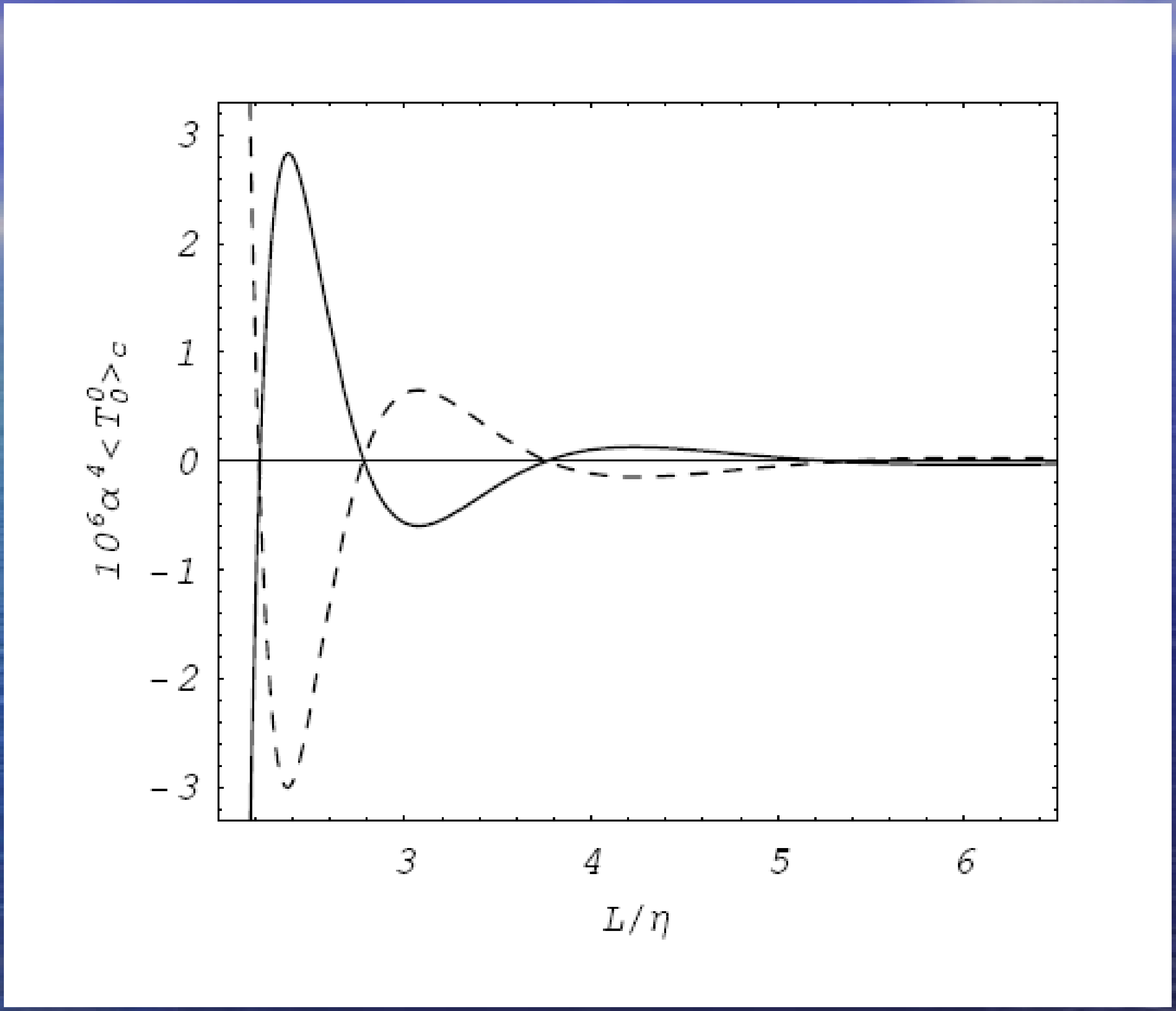
Late stages of the cosmological evolution, $t \to +\infty$



Uncompactified dS part dominates

Topological part is damping oscillatory

Topological part of the vacuum energy density



The topological parts in the VEV of the energy density for periodic (dashed curve) and antiperiodic (full curve) spinor fields in dS spacetime with spatial topology $R^2 \times S^1$ for $\alpha m =$ ٤

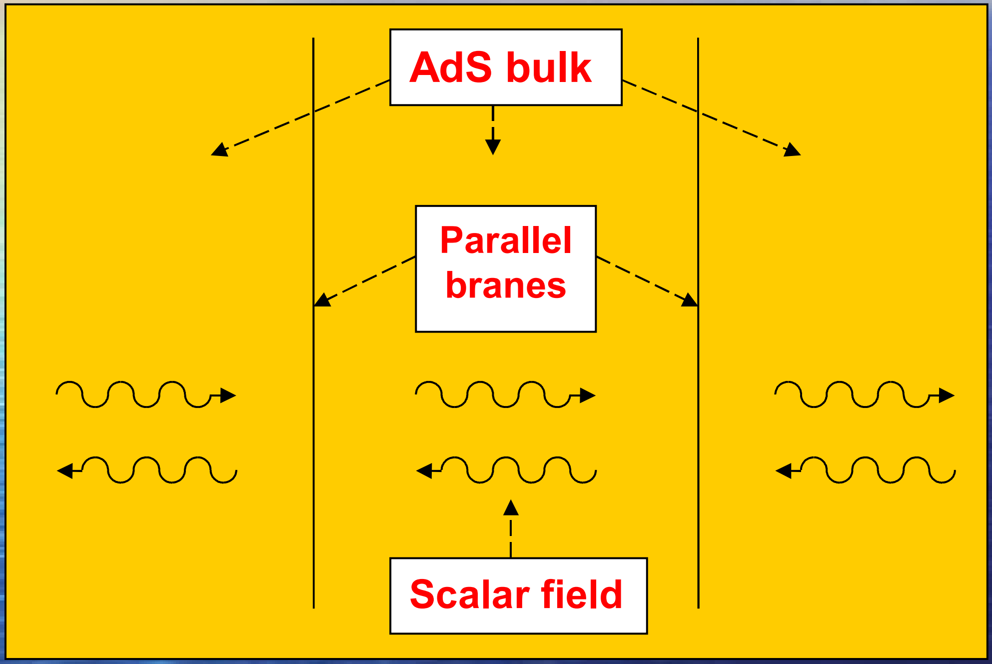
Quantum effects in braneworlds

- Quantum effects in braneworld models are of considerable phenomenological interest both in particle physics and in cosmology
- Braneworld corresponds to a manifold with boundaries and all fields which propagate in the bulk will give Casimir type contributions to the vacuum energy and stresses
- Vacuum forces acting on the branes can either stabilize or destabilize the braneworld
- Casimir energy gives a contribution to both the brane and bulk cosmological constants and has to be taken into account in the self-consistent formulation of the braneworld dynamics

Vacuum densities in braneworlds

1. A. A. Saharian, M. R. Setare, *Phys. Lett.* B584, 306 (2004).
2. A. A. Saharian, *Nucl. Phys.* B712, 196 (2005).
3. A. A. Saharian, *Phys. Rev.* D70, 064026 (2004).
4. A. A. Saharian, M. R. Setare, *Nucl. Phys.* B724, 406 (2005).
5. A. A. Saharian, *Phys. Rev.* D73, 044012 (2006).
6. A. A. Saharian, *Phys. Rev.* D73, 064019 (2006).
7. A. A. Saharian, M. R. Setare, *Phys. Lett.* B637, 5 (2006).
8. A. A. Saharian, *Phys. Rev.* D74, 124009 (2006).
9. A. A. Saharian, M. R. Setare, *JHEP* 02, 089 (2007).
10. A. A. Saharian, A. L. Mkhitaryan, *JHEP* 08, 063 (2007).
11. A. A. Saharian, A. L. Mkhitaryan, *J. Phys. A: Math. Theor.* 41, 164062 (2008).

Geometry of the problem



Bulk geometry: D+1-dim AdS spacetime

$$ds^2 = e^{-2k_D y}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2$$
$$= (k_D z)^{-2}\eta_{ik}dx^i dx^k,$$
$$x^D = z = e^{k_D y}/k_D$$

Branes: Minkowskian branes $R^{(D,1)}$ located at $y=a$ and $y=b$

Field and boundary conditions

Field: Scalar field with an arbitrary curvature coupling parameter

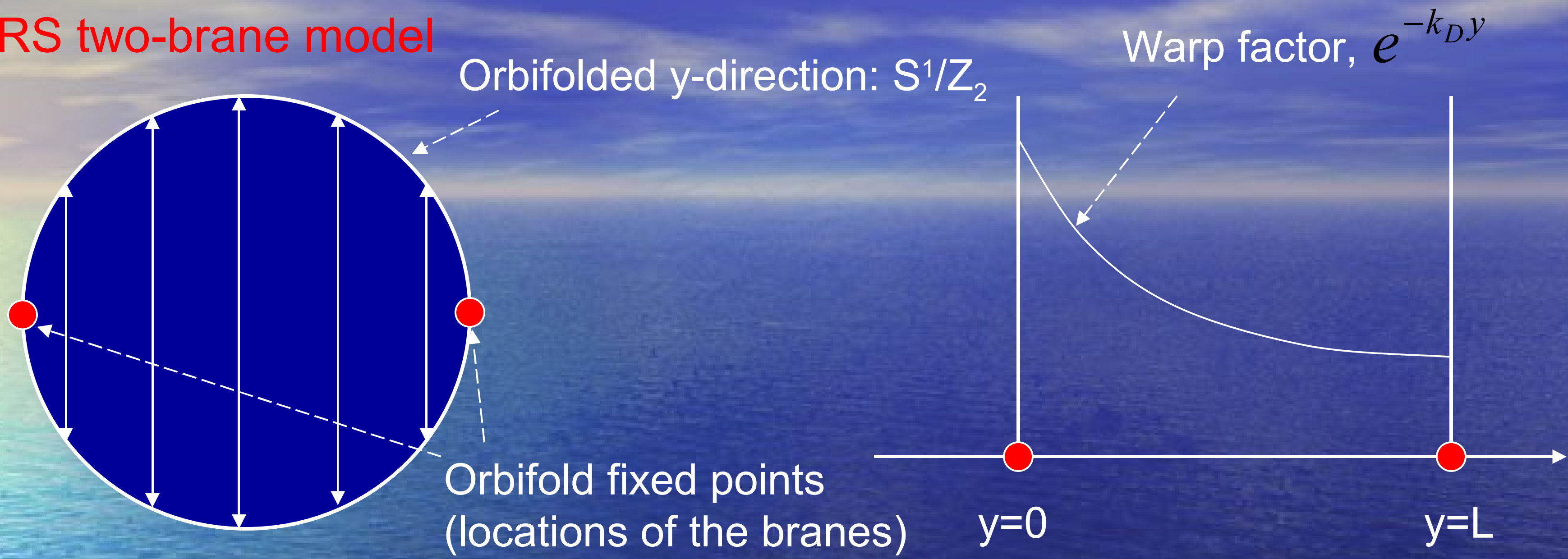
$$(\nabla_i \nabla^i + m^2 + \zeta R)\varphi = 0$$

Boundary conditions on the branes:

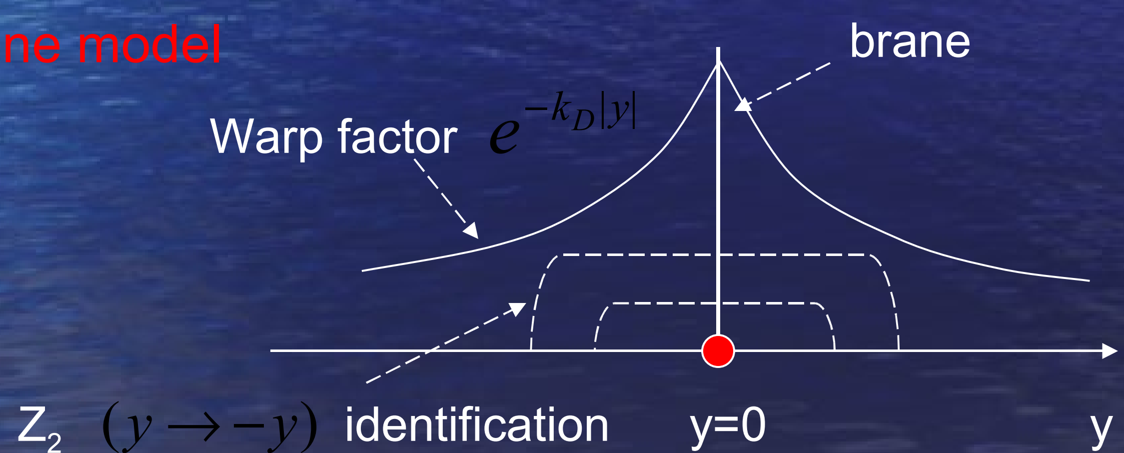
$$(\tilde{A}_y + \tilde{B}_y \partial_y)\varphi(x) = 0, \quad y = a, b$$

Randall-Sundrum braneworld models

RS two-brane model



RS single brane model



Wightman function

- Comprehensive insight into vacuum fluctuations is given by the Wightman function

$$W(x,x') = \langle \cdot | \varphi(x)\varphi(x') | \cdot \rangle = \sum_{\alpha} \varphi_{\alpha}(x)\varphi^{*}_{\alpha}(x')$$

Complete set of solutions to the field equation

- Vacuum expectation values (VEVs) of the field square and the energy-momentum tensor

$$\langle 0 | \varphi^{2}(x) | 0 \rangle = \lim_{x' \to x} W(x,x')$$

$$\langle \cdot | T_{ik}(x) | \cdot \rangle = \lim_{x' \to x} \partial_{i}\partial'_{k} W(x,x') + \left[\left(\zeta - \frac{1}{4} \right) g_{ik} \nabla_{l} \nabla^{l} - \zeta \nabla_{i} \nabla_{k} - \zeta R_{ik} \right] \langle \cdot | \varphi^{2}(x) | \cdot \rangle$$

- Wightman function determines the response of a particle detector of the Unruh-deWitt type

Bulk stress tensor

- Vacuum energy-momentum tensor in the bulk

$$\langle \cdot | T_{MN} | \cdot \rangle = \lim_{x' \to x} \hat{D}_{MN} G^{+}(x, x')$$

← second order differential operator

- Decomposition of the EMT

$$\langle 0 | T_{MN} | 0 \rangle = \langle T_{MN} \rangle_0$$
+ Part induced by a single brane at *y=a* + Part induced by a single brane at *y=b*
+ Interference part

- Vacuum EMT is diagonal

$$\langle 0 | T_{MN} | 0 \rangle = \mathrm{diag}(\varepsilon, \ldots, -p_{\|}, \ldots, -p_{\perp})$$

Vacuum energy density

Vacuum pressures in directions parallel to the branes

Vacuum pressure perpendicular to the branes

Vacuum forces acting on the branes

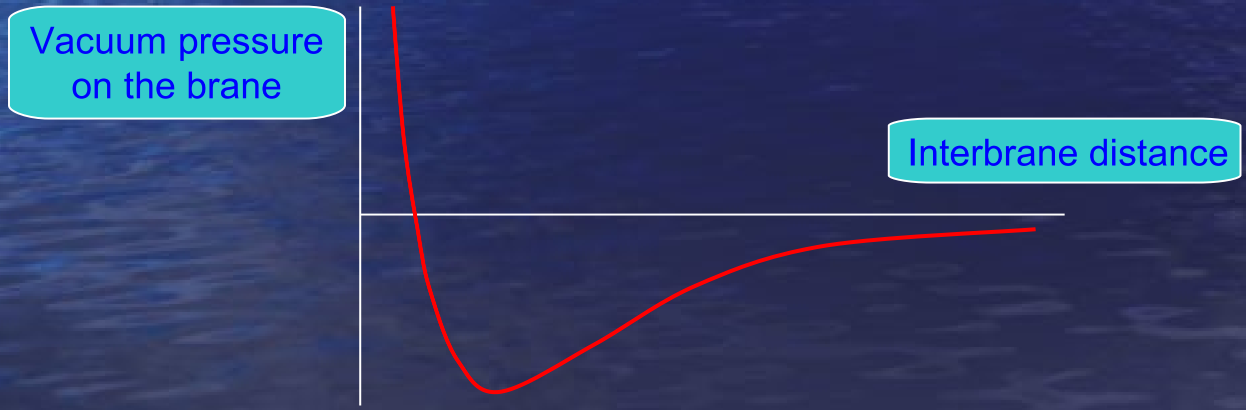
- Vacuum force acting per unit surface of the brane

= Self-action force + Force acting on the brane due to the presence of the second brane (Interaction force)

- In dependence of the coefficients in the boundary conditions the vacuum interaction forces between the branes can be either attractive or repulsive

In particular, the vacuum forces can be repulsive for small distances and attractive for large distances

Stabilization of the interbrane distance (radion field) by vacuum forces



Surface energy-momentum tensor

- Total vacuum energy per unit coordinate surface on the brane

$$E = \frac{1}{2}\sum_{\alpha}\omega_{\alpha}$$

- Volume energy in the bulk

$$E^{(v)} = \int d^{D+1}x\sqrt{|g|}\langle 0|T_0^{(v)0}|0\rangle$$

$$E^{(v)} \neq E$$

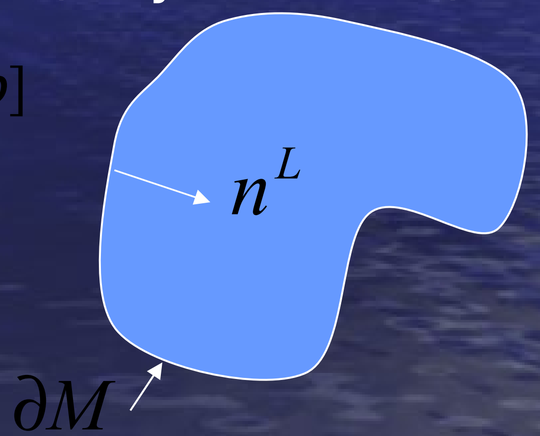
- In general
- Difference is due to the presence of the surface energy located on the boundary
- For a scalar field on manifolds with boundaries the energy-momentum tensor in addition to the bulk part contains a contribution located on the boundary **(A.A.S., Phys. Rev. D69, 085005, 2004)**

$$T_{MN}^{(s)} = \delta(x;\partial M)[\zeta\varphi^2 K_{MN} - (2\zeta - 1/2)h_{MN}\varphi n^L \nabla_L \varphi]$$

Extrinsic curvature tensor

Induced metric

Unit normal



Induced cosmological constant

- Vacuum expectation value of the surface EMT on the brane at *y=j*

$$\langle 0 | T_M^{(s)N} | 0 \rangle = \mathrm{diag}(\varepsilon_j^{(s)}, ..., -p_j^{(s)}, ...), \quad \varepsilon_j^{(s)} = -p_j^{(s)}$$

- This corresponds to the generation of the cosmological constant on the branes by quantum effects

- Induced cosmological constant is a function of the interbrane distance, AdS curvature radius, and of the coefficients in the boundary conditions:

- In dependence of these parameters the induced cosmological constant can be either positive or negative

Physics for an observer on the brane

- *D*-dimensional Newton's constant G_{Dj} measured by an observer on the brane at *y=j* is related to the fundamental *(D+1)*-dimensional Newton's constant $G_{D+١}$ by the formula

$$G_{Dj} = \frac{(D-2)k_D G_{D+1}}{e^{(D-2)k_D(b-a)} - 1} e^{(D-2)k_D(b-j)}$$

- For large interbrane distances the gravitational interactions on the brane *y=b* are exponentially suppressed. This feature is used in the Randall-Sundrum model to address the hierarchy problem
- Same mechanism also allows to obtain a naturally small cosmological constant on the brane generated by vacuum fluctuations

$$\Lambda_{Dj} = ٨\pi G_{Dj}\varepsilon_j^{(s)} \sim ٨\pi G_{Dj} M_{Dj}^{D} e^{-(D+٢\nu)k_D(b-a)}$$

Cosmological constant on the brane *y=j*

Effective Planck mass on the brane *y=j*

$$\nu = \sqrt{D^2/4 - D(D+1)\zeta + m^2/k_D^2}$$

Induced cosmological constant in Randall-Sundrum model

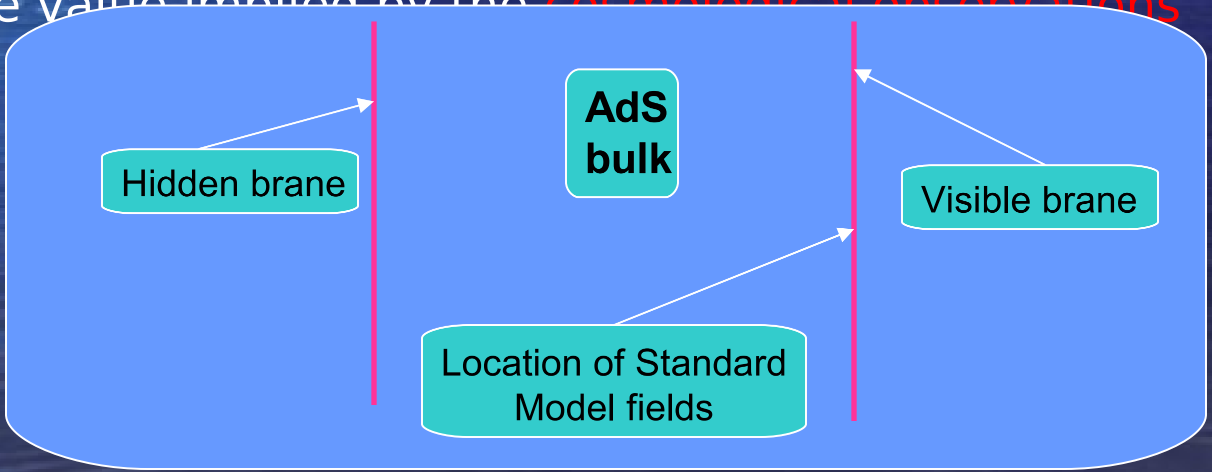
- For the Randall-Sundrum brane model D=4, the brane at $y=a$ corresponds to the hidden brane and the brane at $y=b$ corresponds to the visible brane

$$M_{D+1} \sim TeV, \quad M_{Db} = M_{Pl} \sim 10^{16} TeV$$

- Observed hierarchy between the gravitational and electrowek scales is obtained for

$$k_D (b-a) \approx 40$$

- For this value of the interbrane distance the cosmological constant induced on the visible brane is of the right order of magnitude with the value implied by the cosmological observations



Conclusion

- There are difficulties to realizing a phenomeno-logically viable version of the scenario, but it is intriguing to note that Casimir energies have the potential of fruitfully linking the issues of
- moduli stabilization
- large/small dimensions
- dark energy
- cosmological constant